$S$, the mass of the balloon skin, is known. $M$ can be computed by adding $S$ to the mass of the air inside the balloon. This latter quantity can be approximated by multiplying the density of air (which the students need to find) by the volume of the balloon. The balloon's volume is approximately that of an oblate ellipsoid of revolution, $\frac{4}{3}\pi a^2 b$, where $a$ and $b$ are the half lengths of the larger axis and the smaller axis of the generating ellipse, respectively. The value for $g$ is 9.81 m/sec$^2$. The only unknown parameter is $k$ in models (ii) and (iii).

At this point a graphical comparison can be made. The points $(T_1, 1)$ and $(T_2, 2)$ are plotted along with a graph of $s(t)$ from model (i). Then $s(t)$ from models (ii) and (iii) are graphed for various values of $k$. To reach a qualitative conclusion, the students choose the model that seems to fit the two data points the "best."

Although the experiment is performed in the classroom, each group has up to two weeks to hand in a written report describing its solution process as well as its results. I distribute a set of questions to guide the solution process. Also, each group member completes an evaluation of each member's participation in the project.

**Variations.** This project can be modified in several ways. I had the students solve the differential equations analytically and then produce graphs of the solutions. Finding the solution for model (iii) proved difficult for most groups in my calculus course, even those that used the computer. Asking for numerical solutions would add an interesting twist. All groups estimated their $k$ values by trial and error, defining the "best" $k$ value qualitatively. Finding the "best" $k$ value in the least squares sense is a one-parameter minimization problem. Questions can be created to lead students in this direction. Finally, other models of resistance can be explored, such as $F(v) = kv^m$, for values of $m$ other than 1 or 2.

*Acknowledgment.* The author first learned of this experiment from an MAA minicourse at the annual meeting in January 1989, "Applied Mathematics via Classroom Experiments," given by Herb Bailey from Rose-Hulman Institute of Technology.

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## Matrix Patterns and Undetermined Coefficients

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A vexing but traditional part of an elementary differential equations course concerns nonhomogeneous linear equations with constant coefficients. I shall describe a simple idea that reinforces the important concept of linearity and clarifies the nature of solutions of such equations that have a simple nonhomogeneous term. Using linear algebra as a shovel, we will dig for useful patterns.

How would you talk about finding a particular solution of this equation with your students?

$$\frac{d^2y}{dt^2} - y = 4te^{at} \tag{1}$$

Traditional approaches include appeals to the Laplace transform, variation of parameters, or the method of undetermined coefficients. This last method often simply amounts to consulting a special table; others justify it by using annihilator operators. We will show how a problem like this one provides an early opportunity to introduce matrix methods into the differential course. The idea is to make use of several patterns associated with matrix multiplication and linear operators.

First, we view the differential equation as an operator equation $L(y) = 4te^{at}$, where $L = D^2 - 1$, $D$ being the differentiation operator $d/dt$. Note the important identity

$$L(e^{at}) = p(a)e^{at}, \quad \text{where } p(x) = x^2 - 1. \tag{2}$$

Polynomials can be considered matrix products, for example

$$4te^{at} = [0 \quad 4]\begin{bmatrix} 1 \\ t \end{bmatrix}e^{at}.$$

Our strategy is to differentiate identity (2) with respect to the parameter $a$, noting that

$$\frac{\partial^2}{\partial t \partial a} = \frac{\partial^2}{\partial a \partial t}, \quad \text{or} \quad D\frac{\partial}{\partial a} = \frac{\partial}{\partial a}D, \quad \text{so that} \quad L\frac{\partial}{\partial a} = \frac{\partial}{\partial a}L.$$

The result is $L(te^{at}) = \{p(a)t + p'(a)\}e^{at}$, which we can combine with (2) in matrix form as

$$L\left(\begin{bmatrix} 1 \\ t \end{bmatrix}e^{at}\right) = \begin{bmatrix} p(a) & 0 \\ p'(a) & p(a) \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix}e^{at}. \tag{3}$$

Compare this with what we want:

$$L(y_p) = 4te^{at} = [0 \quad 4]\begin{bmatrix} 1 \\ t \end{bmatrix}e^{at}.$$

By the linearity of $L$, $y_p = [c_0 \quad c_1]\begin{bmatrix} 1 \\ t \end{bmatrix}e^{at}$ will be a solution exactly when

$$[c_0 \quad c_1]\begin{bmatrix} p(a) & 0 \\ p'(a) & p(a) \end{bmatrix} = [0 \quad 4]. \tag{4}$$

Therefore, since

$$\begin{bmatrix} p(a) & 0 \\ p'(a) & p(a) \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{p(a)} & 0 \\ -\dfrac{p'(a)}{p(a)^2} & \dfrac{1}{p(a)} \end{bmatrix},$$

then

$$[c_0 \quad c_1] = [0 \quad 4]\begin{bmatrix} \dfrac{1}{p(a)} & 0 \\ -\dfrac{p'(a)}{p(a)^2} & \dfrac{1}{p(a)} \end{bmatrix} = \left[ -\frac{4p'(a)}{p(a)^2} \quad \frac{4}{p(a)} \right]. \tag{5}$$

We conclude that a particular solution of (1) is

$$y_p = \left\{ -\frac{4p'(a)}{p(a)^2} + \frac{4}{p(a)}t \right\} e^{at}, \qquad \text{provided } p(a) \neq 0,$$

or

$$y_p = \frac{4}{(a^2-1)^2}\{-2a + (a^2-1)t\}e^{at}. \tag{6}$$

The solution structure changes when $p(a) = 0$, and we will return to this case later.

The representation (6) contains more than might first be apparent. For example, to solve $(d^2y/dt^2) - y = 4t$ we would simply set $a = 0$. To solve $(d^2y/dt^2) - y = 4t\cos(\omega t)$, just set $a = i\omega$ and solve $L(y_p) = [0 \quad 4]\left[\begin{smallmatrix} 1 \\ t \end{smallmatrix}\right] e^{i\omega t}$ as above. Since $L$ is a linear operator, if we write $y_p = u + iv$ then, by Euler's identity, $L(u) + iL(v) = 4t\{\cos(\omega t) + i\sin(\omega t)\}$, and we conclude that a particular solution is the real part of (6):

$$\begin{aligned} y_p &= \frac{4}{(\omega^2+1)^2}\operatorname{Re}\{-2i\omega + (-\omega^2-1)t\}\{\cos(\omega t) + i\sin(\omega t)\} \\ &= \frac{4}{(\omega^2+1)^2}\{2\omega\sin(\omega t) - (\omega^2+1)t\cos(\omega t)\}. \end{aligned}$$

The particular solution (5) is quite general—it applies to any differential equation $L(y) = 4te^{at}$ with constant coefficients, when the characteristic polynomial $p(x)$ does not vanish at $x = a$. The order of the equation does not matter. The degree of the polynomial that occurs on the right side of the equation determines the size of the matrix involved. If this term were a fourth degree polynomial times $e^{at}$, say $(q_0 + q_1 t + \cdots + q_4 t^4)e^{at}$, equation (4) would become

$$\begin{aligned} [c_0 \quad c_1 \quad \ldots \quad c_4] &\begin{bmatrix} p(a) & 0 & 0 & 0 & 0 \\ p'(a) & p(a) & 0 & 0 & 0 \\ p''(a) & 2p'(a) & p(a) & 0 & 0 \\ p^{(3)}(a) & 3p''(a) & 3p'(a) & p(a) & 0 \\ p^{(4)}(a) & 4p^{(3)}(a) & 6p''(a) & 4p'(a) & p(a) \end{bmatrix} \\ &= [q_0 \quad q_1 \quad \ldots \quad q_4] \end{aligned}$$

and the triangular matrix could be easily inverted, provided $p(a) \neq 0$. Note the occurrence of the binomial coefficients here. A nice project for a motivated class is to find a general formula for the inverses of such Pascal matrices [**2**].

The structure of the solution changes if $p(a) = 0$, i.e., when $e^{at}$ is a solution of the homogeneous equation $L(y) = 0$. But our strategy remains largely the same. We differentiate (2) *twice* with respect to the parameter $a$, and group the results in matrix form:

$$L\left(\begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} e^{at}\right) = \begin{bmatrix} p(a) & 0 & 0 \\ p'(a) & p(a) & 0 \\ p''(a) & 2p'(a) & p(a) \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} e^{at}.$$

The first row of the matrix is zero, so the linear system reduces to

$$L\left(\begin{bmatrix} t \\ t^2 \end{bmatrix} e^{at}\right) = \begin{bmatrix} p'(a) & 0 \\ p''(a) & 2p'(a) \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} e^{at}.$$

Thus

$$y_p = [c_0 \quad c_1] \begin{bmatrix} t \\ t^2 \end{bmatrix} e^{at}$$

will be a particular solution of $L(y) = 4te^{at}$ exactly when the row vector $[c_0 \quad c_1]$ satisfies

$$[c_0 \quad c_1] \begin{bmatrix} p'(a) & 0 \\ p''(a) & 2p'(a) \end{bmatrix} = [0 \quad 4].$$

For example if $p(x) = x^2 - 1$ as before and $a = 1$, we have

$$[c_0 \quad c_1] \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix} = [0 \quad 4],$$

so

$$[c_0 \quad c_1] = [0 \quad 4] \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = [-1 \quad 1].$$

That is, a particular solution of $(d^2y/dt^2) - y = 4te^t$ is $y_p = (-t + t^2)e^t$.

The method we have outlined above has several advantages over the usual textbook approaches to the method of undetermined coefficients:

- Solutions of the homogeneous equation are not required in advance, and the complete root structure of the characteristic polynomial is not needed. We simply differentiate identity (2) $k+1$ times with respect to the parameter $a$, where $k$ is the multiplicity of $a$ as a root of the characteristic polynomial $p(x)$.
- No annihilator operator or Laplace transform techniques are required. Coddington came close to developing this method while validating the annihilator method [**3**], but he did not point out the matrix patterns introduced here.
- The interaction between the operator and the form of the right side of the equation is clarified. The operator information is contained in the characteristic polynomial, and the nonhomogeneous term of the equation determines the matrix to be inverted, according to whether or not the exponential factor in this term is a root of the characteristic polynomial.

- The matrix method produces the solution in a compact form. The solutions produced by computer algebra systems do not offer comparable insight into the structure of the solution without some postprocessing. (Matrix systems of differential and difference equations are discussed in a little book by LaSalle [**4**] that should be read more widely.)
- Our method reinforces the concepts of linearity, matrix multiplication, and the Leibniz rule for differentiating a product, and it provides practice with complex numbers. These are all important topics in science and engineering. The techniques presented here also apply to linear difference equations.
- The idea of differentiating an identity with respect to a parameter, which used to be common in advanced calculus texts, has many applications [**1**]. Thus our method gives students an opportunity to apply important mathematical ideas to solve a class of problems that in the past has often served only to convince students that the introductory differential equations course consists mainly of drill in elementary algebra and calculus.

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## The Lighter Side of Differential Equations


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Although differential equations have many serious applications to the modeling of real-world problems, a few lighthearted problems can serve to motivate students and brighten their attitudes toward a computer-oriented course in differential equations. The following two scenarios are uninhibited by reality.

The first problem involves a system of two coupled linear differential equations, which model the ups and downs of a love affair between Romeo and Juliet. In searching for the origins of the basic idea for this problem, we backtracked along an interesting trail and traced the source to Steven Strogatz of MIT. He contributed the problem to a Harvard final examination, although he had originated it during his college days (perhaps when Romeo–Juliet interactions were more compelling). He later wrote a brief article for *Mathematics Magazine* [**7**], and his use of the problem stimulated a column in 1988 by Clarence Peterson in the *Chicago Tribune*, "As Usual, Boy + Girl = Confusion" [**5**]. More recently, Michael Radzicki of Worcester Polytechnic Institute described using a general version of the problem to teach system dynamics skills [**6**]. The problem has surfaced with many variations and is now passing into the folklore. We hope that the following variations are amusing (and original). The lab exercises described were very popular with students at Cornell and Cal Poly and contributed more than any others to the students' understanding of the relationships among the $xy$, $tx$, and $ty$ graphs.