$(x^n)' = nx^{n-1}$: Six Proofs

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A perusal of calculus textbooks published during the past twenty years reveals six distinct approaches to the proof that the derivative of x^n is $(x^n)' = nx^{n-1}$ for a positive integer n. These proofs use important techniques that should be in the repertoire of every calculus student.

I. Proof with induction and the product rule. First, $(x)' = 1$ by definition. Assuming that $(x^{n-1})' = (n-1)x^{n-2}$ and using the product rule we have

$$(x^n)' = (x^{n-1}x)' = (x^{n-1})'x + x^{n-1}(x)' = nx^{n-1}.$$

[L. Bers with F. Karal, *Calculus*, Holt, Rinehart, and Winston, 1976, p. 87; N. Friedman, *Basic Calculus*, Scott, Foresman, 1968, p. 112; S. Salas and E. Hille, *Calculus: One and Several Variables with Analytic Geometry*, Wiley, 1974, p. 84]

II. Proof using properties of logarithms. If $y = x^n$, $\ln y = n \ln x$ and logarithmic differentiation yields $y'/y = n/x$ which immediately implies $y' = nx^{n-1}$. [L. Hoffmann, *Calculus for Business, Economics, and the Social and Life Sciences*, McGraw-Hill, 1986, p. 255]

III. Proof by estimation. We use the big O notation common in analytic proofs: We say $f(h) = O(g(h))$ if there is some constant C such that, if h is sufficiently close to 0, then $f(h) < C|g(h)|$. This is slightly nonstandard (h goes to zero instead of infinity) but it clarifies the proof.

Lemma. $(x+h)^n = x^n + nx^{n-1}h + O(h^2)$.

The lemma has a direct inductive proof independent of the binomial expansion. The lemma is used to simplify the numerator of the difference quotient $[(x+h)^n - x^n]/h$. A similar simplification occurs in proofs IV and V. [S. Stein, *Calculus and Analytic Geometry*, McGraw-Hill, 1987, p. 85]—Stein, however, does not use the big O notation.

IV. Proof using factoring. We use the factor formula for the difference of nth powers, $(x+h)^n - x^n = (x+h-x)\Sigma(x+h)^{n-1-i}x^i$, to simplify the difference quotient. [S. Grossman, *Calculus (International Edition)*, Academic Press, 1981, p. 135; L. Loomis, *Calculus*, Addison-Wesley, 1974, p. 100; A. Spitzbart, *Calculus with Analytic Geometry*, Scott, Foresman, 1975, pp. 85–86]

V. Proof using the binomial theorem. We use the binomial expansion, $(x+h)^n = \Sigma\binom{n}{i}x^{n-i}h^i$, to simplify the difference quotient. This seems to be the most popular method. [H. Anton, *Calculus with Analytic Geometry*, Wiley, 1988, pp. 159–160; J. Fraleigh, *Calculus with Analytic Geometry*, Addison-Wesley, 1980, p. 45; L. Leithold, *The Calculus with Analytic Geometry*, Harper and Row, 1986, p. 189; G. Simmons, *Calculus with Analytic Geometry*, McGraw-Hill, 1985, pp. 63–64; E. Swokowski, *Calculus with Analytic Geometry*, Prindle, Weber and Schmidt, 1983, pp. 99–100; A. Willcox, R. Buck, H. Jacob, and D. Bailey, *Introduction to Calculus 1 and 2*, Houghton Mifflin, 1971, pp. 63–64]

VI. "Proof" by example. Finally, not every formula is totally proven in math courses. In such cases, computation of several examples or proof of several subcases of the main theorem is a welcome procedure [L. Goldstein, D. Lay, and D. Schneider, *Calculus and Its Applications*, Prentice-Hall, 1980, pp. 57–61]. Verification by examples or subcases has intrinsic value even when a general proof is given. For example, some calculus students can prove $(x^n)' = nx^{n-1}$ by the binomial theorem for general n, but still fumble when asked to prove it for the case $n = 2$ or $n = 3$.

Occasionally, a book does present several of the above proofs [H. Flanders and J. Price, *Calculus with Analytic Geometry*, Academic Press, 1978, p. 65; L. Bers, *Calculus*, Holt, Rinehart and Winston, 1969, p. 169]. Some proofs (II, III) are mentioned rarely. Even proof I, despite the importance of induction, occurs in only about 20% of the books. I suggest that freshman calculus courses be enriched by presenting all the above proofs.

Measurable Metaphors

What does a millimeter of pizazz more or less matter when we are discussing the nth degree of wonderful?

Thomas M. Disch, Jerome Robbins' Broadway in circles, *The Nation* (May 1989) 713.

Did You Know?

Parabolas (Gr., thrown the same) are formed by "throwing" a plane through a cone so that the plane makes the "same" angle with the base as does the side of the cone. Similarly, the angle made with the base by the ellipse (Gr., less than, wanting) is "less than" that made by the side and the angle made by the hyperbola (Gr., thrown greater) is "greater."