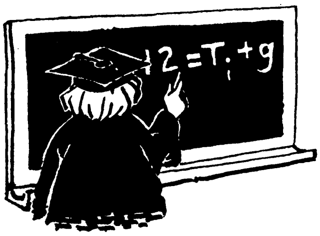

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

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Some Subtleties in L'Hôpital's Rule

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The use of L'Hôpital's Rule (actually a theorem of Johann Bernoulli) to calculate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$, for example, is almost always presented as a simple string of equalities:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2} \tag{1}$$

$$=\lim_{x\to 0}\frac{\sin x}{6x} \tag{2}$$

$$=\lim_{x\to 0}\frac{\cos x}{6} \tag{3}$$

$$=\frac{1}{6}.$$

Of course, the first three equalities are not justified until the last equality has been reached. A more complete presentation would add, "if this limit exists" after the second, third, and fourth limits (a limit of $\pm\infty$ is considered here to "exist"), and the conclusion would be: "Since $\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$, Equation (3) holds; therefore Equation (2) holds; therefore Equation (1) holds; hence $\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1}{6}$."

We do not advocate such a complete presentation for every example of L'Hôpital's Rule, but we do suggest that students be made aware of the importance of the existence hypothesis. Although the existence of $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ is a sufficient condition for the existence of $\lim_{x\to a}\dfrac{f(x)}{g(x)}$, very little is usually said about whether or not this is a necessary condition for the existence of $\lim_{x\to a}\dfrac{f(x)}{g(x)}$. Thus, it may be enticing for students to believe that when a chain of "L'Hôpital equalities" leads to a limit that does *not* exist, then the original limit also does not exist. The following examples show that such a conclusion may or may not hold.

Example 1 ($\frac{0}{0}$ indeterminate forms). If $f(x) = x^2\sin(x^{-1})$ and $g(x) = \sin x$, then $\lim_{x\to 0}\dfrac{f'(x)}{g'(x)}$ does not exist, whereas $\lim_{x\to 0}\dfrac{f(x)}{g(x)} = 0$. On the other hand, if $f(x) = x\sin(x^{-1})$ and $g(x) = \sin x$, then neither $\lim_{x\to 0}\dfrac{f'(x)}{g'(x)}$ nor $\lim_{x\to 0}\dfrac{f(x)}{g(x)}$ exists.

Example 2 ($\frac{\infty}{\infty}$ indeterminate forms). If $f(x) = x(2 + \sin x)$ and $g(x) = x^2 + 1$, then $\lim_{x\to\infty}\dfrac{f'(x)}{g'(x)}$ does not exist, whereas $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = 0$. On the other hand, if $f(x) = x(2 + \sin x)$ and $g(x) = x + 1$, then neither $\lim_{x\to\infty}\dfrac{f'(x)}{g'(x)}$ nor $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$ exists.

Editor's Note: For a variation on this theme, see J. P. King's Classroom Capsule "L'Hôpital's Rule and the Continuity of the Derivative," TYCMJ 10 (June 1979), 197–198.

An Analytic Approach to the Euler Line

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In the November 1982 Classroom Capsules Column, Norman Schaumberger presented a demonstration of the Euler line using vector methods. It may be interesting to supplement Schaumberger's argument, which has a distinct geometric flavor, with the following analytic approach.

For any triangle ABC, let $G = \frac{1}{3}(A + B + C)$. Then G must lie on all three of the triangle's medians—that is, G is the centroid of triangle ABC.