# CLASSROOM CAPSULES

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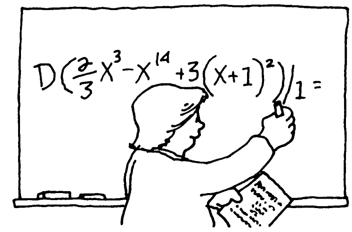


A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

## Ways of Looking at *n*!

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In his note "Another Way of Looking at $n!$" [*Two-Year College Mathematics Journal* 11 (1980) 333–334] David Hsu observed that, as well as representing the number of linear permutations on $n$ symbols, $n!$ may also be interpreted as the reciprocal of the volume of a simplex whose vertices are located at $n+1$ of the corners of the unit cube in Euclidean $n$-space. In this note we verify the connection between the two interpretations. Our purpose is not to present something really new but to draw attention to a point of view that could escape notice.

For notational simplicity we will consider the case where $n = 3$. Suppose three numbers are chosen at random from the interval $(0,1)$. Let $x$ be the first number chosen, $y$ the second and $z$ the third. The probability that these three numbers have the ordering of magnitude $x < y < z$ is $1/3!$ since there are $3!$ permutations of the three letters, each corresponding to an equally probable ordering.

We now regard $x$, $y$ and $z$ as coordinates of a point $X$ in 3-dimensional Euclidean space. Let $T$ be the tetrahedron whose vertices are $(0,0,0)$, $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$. Thus $x < y < z$ if and only if $X \in T$, (see Figure 1), and so the probability that $x < y < z$ is the ratio of the volume of $T$ to the volume of the unit cube. But, as previously stated, this probability is $1/3!$, so the volume of $T$ must be $1/3!$.

Notice that we need not depend on the figure to establish that $x < y < z$ is equivalent to $X \in T$. We may write $(x, y, z)$ as the linear combination

$$(1-z)(0,0,0) + (z-y)(0,0,1) + (y-x)(0,1,1) + x(1,1,1).$$

Using the general definition of a simplex in $n$-dimensional space, we know that this linear combination represents a point inside the simplex $T$ if and only if the coefficients add to 1 and are all positive. This is the case if and only if $x < y < z$ (recalling that $x$, $y$ and $z$ are in the interval $(0,1)$).

In the same way we may show that the open simplex in $n$-dimensional Euclidean space defined by $x_1 < x_2 < x_3 < \cdots < x_n$ with $0 < x_i < 1$ has volume $1/n!$.

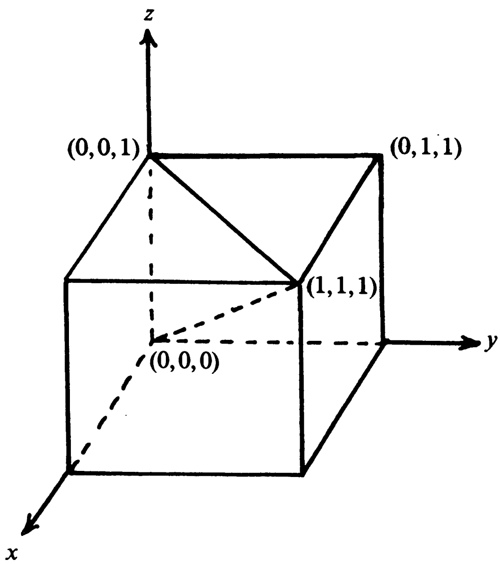


Figure 1

We remark that by permuting the $n$ letters $x_i$ we get $n!$ disjoint regions each of which clearly has the same volume. Thus the unit cube may be dissected into $n!$ simplexes each having the same volume. This statement for the case $n = 3$ will be familiar to those who have taken a traditional high school course in solid geometry.

A recent note of D. J. Smith and M. K. Vamanamurthy [*Mathematics Magazine*, 62 (1989) 101–107] invites some comparisons. The volume $V_n(r)$ of a ball of radius $r$ in $n$-dimensional Euclidean space is $r^n\pi^{n/2}/\Gamma((n/2)+1)$. The authors show that, for a fixed $r$, $V_n(r)$ is monotone decreasing past a certain value of $n$ and tends to zero as $n$ tends to infinity. (In contrast, the volume $c^n$ of a cube with edge $c$ tends to infinity for $c$ a fixed number greater than 1.) If we let $W_n(c)$ be the volume of a simplex with vertices at $n+1$ of the corners of a cube with edge $c$, it follows from our discussion that $W_n(c) = c^n/n!$. It is easily seen that, for fixed $c$, $W_n(c)$ is also monotone decreasing past a certain value of $n$ and tends to zero as $n$ tends to infinity. Another result given in the note mentioned is

$$\sum_{n=0}^{\infty} V_n(r) = e^{\pi r^2}\left(1 + \frac{2}{\sqrt{\pi}}\int_0^{r\sqrt{\pi}} e^{-t^2}\,dt\right).$$

In our case, $\sum_{n=0}^{\infty} W_n(c)$ is also convergent, the sum being $e^c$.

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## Proofs by—Tion

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As a teacher of college mathematics, I've introduced methods of proof to first-year students as well as to upperclassmen taking more advanced courses. These methods are applied to a variety of difficult and important areas of analysis and algebra. Through the years, I've also seen these same students struggle with these profound ideas and I've read all manner of proofs on homework and examinations. While