Alternatively, since $\sinh^{-1}(0) = \tan^{-1}(0) = 0$ and

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}} > \frac{1}{1+z^2} = \frac{d}{dz}\tan^{-1} z \quad \text{for } z > 0,$$

we have $\sinh^{-1} z > \tan^{-1} z$ for $z > 0$, and hence

$$\mathrm{T_d} = \frac{1}{\sqrt{gk}}\sinh^{-1}\left(\sqrt{k/g}\,U\right) > \frac{1}{\sqrt{gk}}\tan^{-1}\left(\sqrt{k/g}\,U\right) = \mathrm{T_u},$$

for $U > 0$.

In Figure 2 we compare the time up and the time down as a function of U in the case with resistance, and also with the time up (and down) U/g in the case with no resistance.

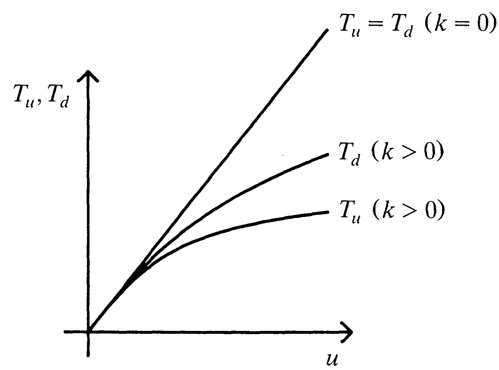

Figure 2

Readers (with or without their classes) may like to investigate this problem in the case where the law of resistance is assumed to be linear instead of quadratic, and possibly to look at a more general form for the law of resistance.

References

1. P. Glaister, Times of flight, *Mathematical Gazette* 74 (1990) 138–139.
2. Murray S. Klamkin, Vertical motion with air resistance, *Proceedings of the 4th International Conference on Mathematical Modelling*, Pergamon, Elmsford, NY, 1984, p. 995.
3. John Lekner, What goes up must come down; will air resistance make it sooner, or later?, *Mathematics Magazine* 55 (1982) 26–28.

Integrals of Products of Sine and Cosine with Different Arguments

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Nearly every calculus text I have encountered in the past several years uses the identities

$$\cos(ax)\cos(bx) = \tfrac{1}{2}\left[\cos((a+b)x) + \cos((a-b)x)\right],$$
$$\sin(ax)\cos(bx) = \tfrac{1}{2}\left[\sin((a+b)x) + \sin((a-b)x)\right],$$
$$\sin(ax)\sin(bx) = \tfrac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right]$$

to evaluate integrals of the form

$$\int \cos(ax)\cos(bx)\,dx,$$

$$\int \sin(ax)\cos(bx)\,dx,$$

$$\int \sin(ax)\sin(bx)\,dx.$$

Most students balk in anticipation of more formulas to memorize.

These integrals are typically found in the section of a text dealing with integrating powers of trigonometric functions, which follows the section on integration by parts. I contend that these integrals should be done by repeated (iterated) integration by parts, just as integrals of the form $\int e^{kx}\cos(ax)\,dx$. Although not so easy as using the above identities, integration by parts is not difficult. For example consider the integral

$$I = \int \sin(2x)\cos(3x)\,dx.$$

Let $u = \sin(2x)$ and $dv = \cos(3x)\,dx$. Then $du = 2\cos(2x)\,dx$, and $v = \frac{1}{3}\sin(3x)$. Thus

$$I = \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\int \cos(2x)\sin(3x)\,dx.$$

Now let $p = \cos(2x)$ and $dq = \sin(3x)\,dx$. Then $dp = -2\sin(2x)\,dx$, and $q = -\frac{1}{3}\cos(3x)$ yielding

$$\begin{aligned}
I &= \tfrac{1}{3}\sin(2x)\sin(3x) - \tfrac{2}{3}\left(-\tfrac{1}{3}\cos(2x)\cos(3x) - \tfrac{2}{3}I\right),\\
I &= \tfrac{1}{3}\sin(2x)\sin(3x) + \tfrac{2}{9}\cos(2x)\cos(3x) + \tfrac{4}{9}I,\\
\tfrac{5}{9}I &= \tfrac{1}{2}\sin(2x)\sin(3x) + \tfrac{2}{9}\cos(2x)\cos(3x) + C.
\end{aligned}$$

Finally,

$$\int \sin(2x)\cos(3x)\,dx = \tfrac{3}{5}\sin(2x)\sin(3x) + \tfrac{2}{5}\cos(2x)\cos(3x) + C.$$

For the student who has been taught tabular integration by parts the calculation runs as follows:

u	dv	
$\sin(2x)$	$\cos(3x)$	$I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}I,$
$2\cos(2x)$	$\frac{1}{3}\sin(3x)$	$\frac{5}{9}I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + C$
$-4\sin(2x)$	$-\frac{1}{9}\cos(3x)$	$I = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.$

The integral is evaluated without the use of trigonometric identities and, as I prefer, in terms of the arguments of the trigonometric functions found in the original problem. As Grant [Moments on a rose petal, *CMJ* (1990) 225–227] mentions, when the result is in terms of the original arguments, checking an integral by differentiation is a viable option, even for the more complex integrals

$\int \sin\theta \sin^n(m\theta)\,d\theta$ and $\int \cos\theta \sin^n(m\theta)\,d\theta$ which Grant tackles using integration by parts. (Incidentally, checking the example above and a few others by differentiation may prompt some to notice the forms that appear as antiderivatives and thereby to sense the possibility of yet another method: undetermined coefficients.)

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A Circular Argument

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Sketch of the circle. The first interesting limit that the student of calculus is exposed to is often

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \qquad (*)$$

This limit has received some attention recently, see [**5**], [**6**], [**9**] and [**13**]. It is not usually recognized that the standard proof is circular, as was suggested in [**13**] and denied in [**9**]. Archimedes proved a variant of $(*)$ in order to show that the area of a circle is equal to the area of a right triangle whose perpendicular sides are the radius and the circumference of the circle. By the definition of π, the circumference of a circle is $2\pi r$, so his theorem establishes the area formula πr^2. Despite this dependence of the area formula on $(*)$, the area formula is the basis for the argument used in most calculus texts to prove $(*)$; see [**3**], [**7**], [**10**], [**11**], [**12**].

The usual proof. The standard argument for $(*)$ hinges on the inequalities

$$\sin x < x < \tan x, \qquad (**)$$

from which $(*)$ readily follows. The usual proof of $(**)$ considers an arc AB of length x on the circle of radius one with center at O, and the point B' on the extension of OB such that AB' is perpendicular to OA. The triangle OAB is contained in the circular sector OAB which is contained in the triangle OAB'. Moreover

(1) the area of the triangle OAB is $(\sin x)/2$,
(2) the area of the circular sector OAB is $x/2$,
(3) the area of the triangle OAB' is $(\tan x)/2$.

Statements (1) and (3) are clearly true; Statement (2) is true because the area of the sector is to the area π of the circle as the length x of the arc AB is to the circumference 2π of the circle.

What's wrong with this proof? The problem lies in how we know that the area of the circle is π. The answer that we learned it in elementary school is not good enough. The fact is that to prove that the area of the circle is π, we have to invoke $(*)$ in some form; for example, in the form of the inequalities $(**)$.

Archimedes' proof that the area of a circle is πr^2. Archimedes was perhaps the first to prove that the area of a circle of radius r is πr^2. Euclid had shown earlier [**4**; XII.2] that the area of a circle is *proportional to* r^2. Archimedes inscribes and circumscribes the circle with regular n-sided polygons. The length of a side of the inscribed polygon is, in our terms, $2\sin \pi/n$, the length of a side of the circumscribed polygon is $2\tan \pi/n$, and $2\pi/n$ is the length of the circular arc between adjacent points of contact of the circle with either polygon.