Proof. To obtain (1) we use the Law of Sines in the five triangles $A_1B_1A_2$, $A_1B_2A_3$, $A_3B_3A_4$, $A_4B_4A_5$, and $A_5B_5A_1$, finding

$$\frac{A_1B_1}{B_1A_2}=\frac{\sin a_2}{\sin a_1'},\quad \frac{A_2B_2}{B_2A_3}=\frac{\sin a_3}{\sin a_2'},\quad \frac{A_3B_3}{B_3A_4}=\frac{\sin a_4}{\sin a_3'},$$
$$\frac{A_4B_4}{B_4A_5}=\frac{\sin a_5}{\sin a_4'},\quad \text{and}\quad \frac{A_5B_5}{B_5A_1}=\frac{\sin a_1}{\sin a_5'},$$

respectively. Since $a_k = a_k'$ for each $k = 1, \ldots, 5$, we can multiply these five equations; (1) is the result. Equation (2) is obtained in a similar way by using the Law of Sines in the five triangles $B_1A_3B_4$, $B_4A_1B_2$, $B_2A_4B_5$, $B_5A_2B_3$, and $B_3A_5B_1$. □

Here is another simple fact about the pentagram that may be surprising and appealing to the beginning student: *The sum of the angles at the points of the star is* $180°$. One way to see this is to observe that $a_1' = \angle B_2 + \angle B_4$, $a_2 = \angle B_3 + \angle B_5$, and $\angle B_1 + a_1' + a_2 = 180°$ and then compute $\angle B_1 + \angle B_2 + \angle B_3 + \angle B_4 + \angle B_5$. This same result can be derived from the fact that the sum of the exterior angles of any convex n-gon is $360°$. Thus, the five triangles containing the points of the star have $a_1 + a_2 + a_3 + a_4 + a_5 = 360°$ and $a_1' + a_2' + a_3' + a_4' + a_5' = 360°$. This leaves $5(180°) - 360° - 360° = 180°$ for $\angle B_1 + \angle B_2 + \angle B_3 + \angle B_4 + \angle B_5$.

———— ○ ————

When Is "Rank" Additive?

David Callan (callan@stat.wisc.edu), University of Wisconsin, Madison, WI 53706

Most matrix theory books mention that rank is subadditive—that is, rank $(A+B) \leq$ rank $(A)+$ rank (B)—but they rarely address the question of equality. Recall that the rank of a matrix A is defined as the *dimension of its column space* $C(A)$. Also, the rank is invariant under transpose: rank (A) = rank (A^T); or, what is the same, the rank of A is the dimension of the row space $R(A)$. (See [**2**] and [**3**] for one-paragraph proofs of this fundamental fact.) This leads to a useful alternative description of the rank: *Rank* (A) *is the size of the largest invertible submatrix of* A.

The subadditivity of rank is easily established: $C(A+B) \subseteq C(A) + C(B)$, hence rank $(A+B) = \dim C(A+B) \leq \dim[C(A)+C(B)] \leq \dim C(A) + \dim C(B) =$ rank $(A)+$ rank (B). Since $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$ for any two subspaces U and V, equality in the second inequality above implies $C(A) \cap C(B) = \{0\}$. Thus disjointness of the column spaces of A and B is a necessary condition for additivity of rank. Curiously, a recent monograph [**4**] asserts incorrectly that this condition is sufficient.

Counterexample. $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $C(A) \cap C(B) = \{0\}$, but rank $(A) =$ rank $(B) =$ rank $(A+B) = 1$).

However another necessary condition is disjointness of the row spaces (since rank is invariant under transpose). It turns out that these two conditions together are sufficient.

Theorem. *Let A and B be $m \times n$ matrices over a field F. Then*

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B),$$

with equality if and only if $C(A) \cap C(B) = \{0\}$ and $R(A) \cap R(B) = \{0\}$.

Proof. It only remains to show the "if" part. By suitable row and column operations, we can reduce A to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $r =$ rank (A) and I_r is the $r \times r$ identity matrix.In other words, $PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ for suitable invertible matrices P and Q—the products of the elementary matrices that perform the row and column operations. Now, pre-multiplication or post-multiplication by an invertible matrix does not affect the rank, so

$$\operatorname{rank}(PAQ) = \operatorname{rank}(A), \quad \operatorname{rank}(PBQ) = \operatorname{rank}(B),$$

and

$$\operatorname{rank}(PAQ + PBQ) = \operatorname{rank}[P(A+B)Q] = \operatorname{rank}(A+B).$$

Also, since invertible linear transformations preserve dimensions of intersections of subspaces, if $C(A) \cap C(B) = \{0\}$ and $R(A) \cap R(B) = \{0\}$, then $C(PAQ) \cap C(PBQ) = \{0\}$ and $R(PAQ) \cap R(PBQ) = \{0\}$. Thus, since both hypotheses and conclusion are unaffected by pre- and post-multiplication by invertible matrices, we may assume that $A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Let $s =$ rank (B) and let U be an $m \times s$ matrix consisting of s linearly independent column vectors that span $C(B)$. Then $B = UV$, where V is the $s \times n$ matrix whose entries in any column are the coefficients in the expression of the corresponding column of B as a linear combination of the columns of U. Clearly U has rank s; so does V, since $R(B) \subseteq R(V)$. Rank $(V) = \dim R(V) \leq s$, the number of rows in V.

Our plan is to exhibit an *invertible* $(r+s) \times (r+s)$ submatrix of $A+B$, which will mean that rank $(A+B) \geq$ rank $(A)+$ rank (B), as required. To this end, partition U and V as $U = \begin{pmatrix} U_r \\ U_p \end{pmatrix}$ and $V = (V_r \; V_q)$, where U_r is $r \times s$ and V_r is $s \times r$. We claim that U_p has independent columns. To see this, suppose $U_p x = 0$ for some vector $x \in F^s$. Then $Ux = \begin{pmatrix} U_r x \\ 0 \end{pmatrix}$ is in $C(A) \cap C(B) = \{0\}$ since, by our special choice of A, $C(A)$ consists of the vectors in F^n all of whose entries after the first r are 0. Thus $Ux = \{0\}$, and since U has independent columns, it follows that $x = 0$. This shows that the columns of U_p are independent; in other words, rank $(U_p) = s$. Hence U_p has an invertible $s \times s$ submatrix U_s whose rows are indexed by a subset J of $\{r+1, \ldots, r+p = m\}$. Similarly, by considering transposes and using the row space hypothesis, the same argument shows that V_q has an invertible $s \times s$ submatrix V_s whose *columns* are indexed by a subset K of $\{r+1, \ldots, r+q = n\}$. Now, the $(r+s) \times (r+s)$ submatrix of B with rows indexed by $\{1, 2, \ldots, r\} \cup J$ and columns indexed by $\{1, 2, \ldots, r\} \cup K$ is $\begin{pmatrix} U_r \\ U_s \end{pmatrix} (V_r \; V_s) = \begin{pmatrix} U_r V_r & U_r V_s \\ U_s V_r & U_s V_s \end{pmatrix}$, so the corresponding submatrix of $A+B$ is $\begin{pmatrix} I_r + U_r V_r & U_r V_s \\ U_s V_r & U_s V_s \end{pmatrix}$. Subtracting $U_r U_s^{-1}$

times the second block row from the first (an operation that does not affect the determinant) gives the matrix $\begin{pmatrix} I_r & 0 \\ U_sV_r & U_sV_s \end{pmatrix}$ with determinant $\det(I_r)\det(U_sV_s) = \det(U_s)\det(V_s) \neq 0$.

We have exhibited an invertible $(r+s)\times(r+s)$ submatrix of $A+B$. Hence rank $(A+B) =$ rank $(A) +$ rank (B), and this proves the theorem.

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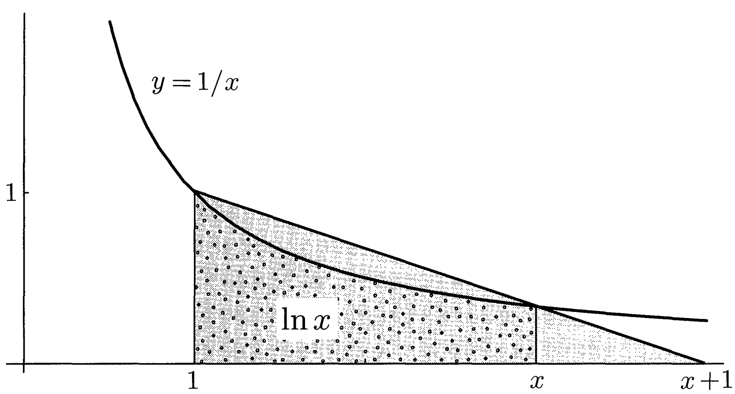
Proof of a Common Limit



From the figure, $\ln x < \frac{1}{2}x$. Thus, $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{x-\ln x}} = 0$.

—*Alan H. Stein and Dennis McGavran*
University of Connecticut