We have found this diagrammatic approach very effective in enforcing the concept of the Laplace transform and its properties, while also providing a visual way of keeping track of the processes used in calculations.

Reference

R. V. Churchill, *Operational Mathematics*, 3rd ed., McGraw Hill, New York, 1972.

A Serendipitous Application of the Pythagorean Triplets

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On occasion, a purely pedagogical consideration leads to an interesting mathematical result. To determine whether students had a good grasp of the process of factoring monic quadratic polynomials, I asked them to factor pairs of the form

$$(x^2+px+q,\ x^2+px-q), \qquad p,q \neq 0 \tag{1}$$

where each polynomial has integer zeros. Examples are:

(i) $(x^2+5x+6,\ x^2+5x-6)$

(ii) $(x^2+13x+30,\ x^2+13x-30)$

(iii) $(x^2+17x+60,\ x^2+17x-60)$.

The natural question arose as to whether it is possible to produce all such pairs of polynomials. As we will see, the answer is yes.

We begin with the observation that once a pair of the desired polynomials is known, an infinite number of pairs can be generated from it; for if the polynomials in (1) have integer zeros, so do $(x^2+tpx+t^2q,\ x^2+tpx-t^2q)$ for each integer t. We are merely producing polynomials whose zeros have been multiplied by t. This motivates the following definitions:

Definitions. A pair of polynomials of the form described in (1) is an *integer quadratic polynomial pair* if each polynomial has integer zeros. If the polynomials in (1) satisfy the additional requirement that $(p,q)=1$, then the polynomials will be called a *representative pair*.

It is readily seen that if $t|(p,q)$ and $(x^2+px+q,\ x^2+px-q)$ is an integer quadratic polynomial pair, then $(x^2+(p/t)x+q/t^2,\ x^2+(p/t)x-q/t^2)$ is also such a pair. It follows that each integer quadratic polynomial pair can be derived from a representative pair and therefore, it suffices to produce all representative pairs. To this end, we prove the following theorem.

Theorem. *The pair $(x^2+px+q,\ x^2+px-q)$ is a representative pair if and only if p and q are of the form*

$$p = \pm(u^2+v^2) \text{ and}$$
$$q = uv(u^2-v^2)$$

in which u and v are integers such that $(u,v)=1$ and u and v are of opposite parity.

Proof. Assume that $(x^2+px+q,\ x^2+px-q)$ is a representative pair. Then the discriminants must be perfect squares. Therefore, there exist integers m and n such that

$$p^2-4q=m^2 \text{ and}$$
$$p^2+4q=n^2$$

from which we obtain by addition

$$2p^2=m^2+n^2 \tag{2}$$

and by subtraction

$$8q=n^2-m^2 \tag{3}$$

Since it is evident that m and n have the same parity, we may change (2) to the equivalent form

$$p^2=[(m+n)/2]^2+[(m-n)/2]^2. \tag{4}$$

We see from (4) that $[(m+n)/2, (m-n)/2, p]$ is a Pythagorean triplet. Our next objective is to show that the triplet is primitive (no common factors greater than 1). To show this, it suffices to prove that $[(m+n)/2, (m-n)/2]=1$. Assume that this is not the case. Then there exists a prime, t, such that $t|[(m+n)/2, (m-n)/2]$. Then t divides both the sum and difference of $(m+n)/2$ and $(m-n)/2$ which implies that $t|m$ and $t|n$. We observe from (2) that m, n and p must each be odd; for m and n cannot be of opposite parity, and if m and n are both even, then the highest power of 2 that divides the left-hand side of (2) is odd while the highest power of 2 that divides the right-hand side of (2) is even unless $m=n$, which is impossible because of (3). Therefore, m and n are both odd, which implies that $m^2+n^2=2(\text{mod}\,8)$. If p were even, then $2p^2=0(\text{mod}\,8)$, a contradiction. Therefore, m, n and p are each odd. Since $t|m$ and m is odd, it follows that t is an odd prime. We also see that $t^2|(m^2+n^2)$. From (2), we can now conclude that $t|p$, and from (3) we see that t also divides q. Therefore $t|(p,q)$ which contradicts the fact that $(x^2+px+q,\ x^2+px-q)$ is a representative pair. This contradiction proves that $[(m+n)/2, (m-n)/2]=1$, which implies that $[(m+n)/2, (m-n)/2, p]$ is a primitive Pythagorean triplet.

All such triplets can be obtained from the known formulae

$$\begin{aligned} p &= u^2+v^2 \\ (m+n)/2 &= 2uv \\ (m-n)/2 &= u^2-v^2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} p &= u^2+v^2 \\ (m-n)/2 &= 2uv \\ (m+n)/2 &= u^2-v^2, \end{aligned}$$

depending on whether $(m+n)/2$ is even or odd, with $(u,v)=1$ and $2|uv$. In either case, by using (3) together with some straightforward algebraic computations, we obtain

$$p = u^2 + v^2 \quad \text{and} \quad q = uv(u^2 - v^2)$$

as specified in the statement of the theorem.

We conclude by noting that examples (i), (ii), and (iii) derive from substituting for (u, v) the ordered pairs (2, 1), (3, 2), and (4, 1), respectively.

Erratum: Relative Maxima or Minima for a Function of Two Variables

The Editors

Several readers, among them Roger Chalkley, Vidhyanath K. Rao, James H. Foster, and Dave Lidstone, have observed that the argument in [Paul Chacon, Relative maxima or minima for a function of two variables: a neglected approach, *College Mathematics Journal* 23 (1992) 145] is based on a false assumption, namely, if the function $f(x, y)$ has a local minimum at the origin when restricted to every straight line through the origin, then this function has a local minimum at the origin. A simple counterexample to the assumption is given by the polynomial

$$f(x, y) = (y - x^2)(y - 2x^2).$$

This or similar examples are discussed and the issue clarified in [**1**], [**2**], [**3**], [**4**], and [**5**]. The same false assumption appears to be made in [**6**] and [**7**], which is anthologized in [**8**].

References

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5. R. Osserman, *Two-Dimensional Calculus*, Harcourt, Brace & World, New York, 1968, 173, example 12.3.
6. A. S. Hendler, Relative maxima and minima of functions of two or more variables, *American Mathematical Monthly* 61 (1954) 418–420.
7. W. C. Stretton, Use of the directional derivative in locating extrema, *Mathematics Teacher*, 63 (1970) 139–142.
8. T. M. Apostol, H. E. Chrestenson, C. S. Ogilvy, D. E. Richmond and N. J. Schoonmaker, (Eds.), *Selected Papers on Calculus*, Mathematical Association of America, Washington, DC, 1969.