CLASSROOM CAPSULES

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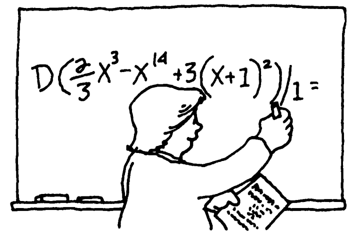

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Halley's Gunnery Rule

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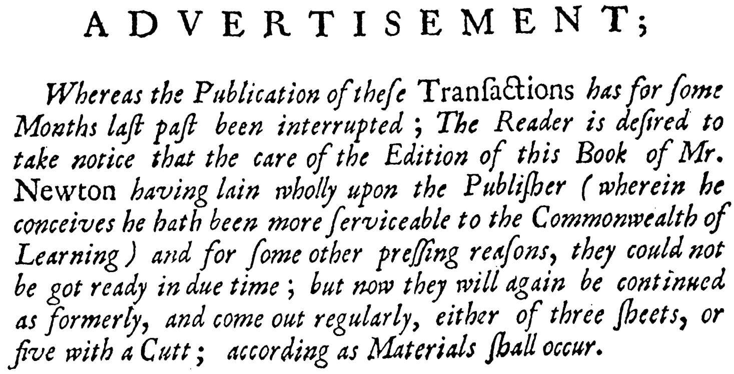
In 1686 Edmond Halley presented a paper [**1**] to the Royal Society in which he summarized the laws governing gravity and the motion of projectiles discovered "by the accurate diligence of Galileus, Torricellius, Hugenius, and others, and now lately by our worthy country-man, Mr Isaac Newton (who has an incomparable *Treatise of Motion* almost ready for the Press)" and applied these principles to derive a theorem in gunnery. At the time, Halley was extremely busy overseeing the printing of Newton's *Principia*; see [**3**]. It is ironic that, while he intended to plug Newton's forthcoming book in his paper, the heavy demands that the *Principia* placed on him caused Halley to neglect his duties as editor of the *Philosophical Transactions* so that the *Transactions* volume containing his paper did not appear until 1688—the year after the publication of Newton's *Principia*. (See Figure 1.)



ADVERTISEMENT;

Whereas the Publication of theſe Tranſactions *has for ſome Months laſt paſt been interrupted ; The Reader is deſired to take notice that the care of the Edition of this Book of Mr.* Newton *having lain wholly upon the Publiſher (wherein he conceives he hath been more ſerviceable to the Commonwealth of Learning) and for ſome other preſſing reaſons, they could not be got ready in due time ; but now they will again be continued as formerly, and come out regularly, either of three ſheets, or five with a Cutt ; according as Materials ſhall occur.*

Figure 1

Halley's paper, like the *Principia*, is phrased in Euclidean language that is very strange to the ears of today's students (and their instructors!). The ideas involved are quite basic, however, and a modern analytic discussion of his gunnery rule provides excellent enrichment material for the calculus classroom.

Before beginning his analysis, Halley framed two simplifying assumptions. Concerning gravity:

> ... considering the smallness of height, to which any *Project* can be made to ascend, and over how little an *Arch* of the *Globe* it can be cast by any of our *Engines*, we may well enough suppose the *Gravity* equal throughout.

As for air resistance:

> ... in great and ponderous Shot, this Impediment is found by *Experience* but very small, and may safely be neglected.

In other words, he takes the force of gravity to be constant and neglects air resistance.

The problem that Halley posed involved firing a projectile through a target situated on an inclined plane (he also considered firing *below* the horizon, i.e., on a declined plane, but we will consider only the situation illustrated in Figure 2). Halley noted later [**2**] that such projectiles when fired with too much force tend to "bury themselves too deep in the ground, to do all the damage that they might ... which is a thing acknowledged by the besieged in all towns, who unpave their streets, to let the bombs bury themselves, and thereby stifle the force of their splinters." Halley therefore asks how the piece should be aimed so as to strike the target using the least powder charge. This charge is proportional to the initial kinetic energy of the ball. Since air resistance is ignored (and hence total energy is conserved), this initial energy, minus the fixed potential energy at the target, is the energy with which the ball strikes the target.

Consider then a target situated at a height h and at a horizontal distance b from the field piece, on a plane inclined at angle α, where $\tan\alpha = h/b$, as in Figure 2. We suppose the piece to be at the origin of the rectangular coordinate system and the angle of elevation of the barrel to be θ. The equations of motion (using dots to indicate time derivatives) are then

$$\ddot{x} = 0, \quad \ddot{y} = -g, \quad \dot{x}(0) = v\cos\theta, \quad \dot{y}(0) = v\sin\theta, \quad x(0) = y(0) = 0,$$

where v is the muzzle velocity and g is the force of gravity (for neatness, we assume the shot has unit mass). Some very straightforward integrations then give

$$x = (v\cos\theta)t,$$
$$y = -\frac{g}{2}t^2 + (v\sin\theta)t.$$

We therefore have a hit on the target if and only if

$$h = -\frac{g}{2}\left(\frac{b}{v\cos\theta}\right)^2 + v\sin\theta\frac{b}{v\cos\theta}$$

or equivalently

$$p(b\tan\theta - h) = b^2(1 + \tan^2\theta) \tag{1}$$

where we have set $p = 2v^2/g$.

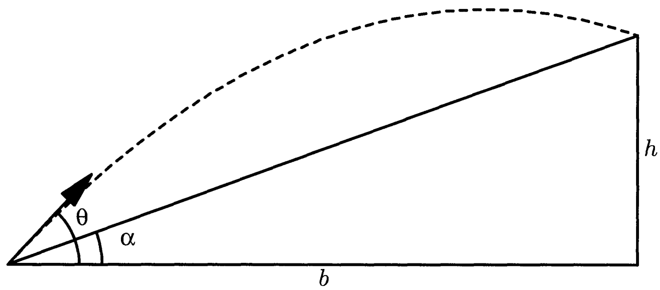


Figure 2

Notice that p is, to within a constant multiple, the initial energy of the ball. Therefore Halley's problem reduces to finding θ so that (1) holds and p is minimized. Setting $u = \tan\theta$, we may rewrite (1) as

$$p = \frac{b^2(1+u^2)}{bu-h}. \tag{2}$$

It is now a routine exercise to find that

$$\frac{dp}{du} = b^2\frac{bu^2-2hu-b}{(bu-h)^2}$$

and

$$\frac{d^2p}{du^2} > 0.$$

Therefore p has a minimum when the numerator vanishes, that is, when

$$u = \frac{h+\sqrt{h^2+b^2}}{b} = \frac{h}{b} + \sqrt{\left(\frac{h}{b}\right)^2 + 1} = a + \sqrt{a^2+1} \tag{3}$$

where $a = \tan\alpha$ (note that $\theta > \alpha$ dictates the positive root).

Nine years later Halley realized that the solution (3) has a tidy geometric interpretation [**2**]. Note that (3) may be expressed in terms of the angles θ and α as $\tan\theta = \tan\alpha + \sec\alpha$. Hence, by a standard trigonometric identity,

$$\begin{aligned}\sec\alpha &= \tan\theta - \tan\alpha\\ &= \frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha}.\end{aligned}$$

Therefore, $\sin(\theta-\alpha) = \cos\theta = \sin(\pi/2-\theta)$, and since both angles are acute,

$$\theta - \alpha = \frac{\pi}{2} - \theta.$$

In other words, *the minimally charged shot strikes the target when the barrel bisects the angle between the inclined plane and the vertical.*

Halley also observed that the aiming (3) not only minimizes the charge, thereby saving gunpowder, but also assures the most accurate shot, in the sense that this aiming is least sensitive to errors in setting the elevation of the weapon. To be more precise, if u is given by (3), then from (1), we have

$$b = \frac{p(u-a)}{1+u^2}. \tag{4}$$

Now b, considered as a variable, and hence h, has minimum sensitivity to small errors $\Delta\theta$ in θ if $\frac{db}{d\theta} = 0$, since $\Delta b \approx \frac{db}{d\theta}\Delta\theta$. However,

$$\frac{db}{d\theta} = \frac{db}{du}\frac{du}{d\theta} = \frac{db}{du}\sec^2\theta,$$

hence $\frac{db}{d\theta} = 0$ if and only if $\frac{db}{du} = 0$. By (4),

$$\frac{db}{du} = \frac{p(1+2au-u^2)}{(1+u^2)^2},$$

from which it follows that when (3) is satisfied,

$$\left.\frac{db}{du}\right|_{u=a+\sqrt{1+a^2}} = 0.$$

In Halley's words [**1**]:

> This Rule may be of good use to all *Bombardiers* and *Gunners*, not only that they may use no more Powder than is necessary, to cast their *Bombs* into the place assigned, but that they may shoot with much more certainty, for that a small Error committed in the *Elevation* of the *Piece*, will produce no sensible difference in the fall of the Shot.

Finally, before the piece is set according to Halley's rule it is necessary to administer the appropriate powder charge. Halley solved this calibration problem as well. His prescription is this: The appropriate charge—i.e., that p satisfying (2), where u is given by (3)—is the same as that which would cast the ball a horizontal distance of $h + \sqrt{h^2 + b^2}$ when the piece is inclined at an angle of 45° to the horizon. We leave the verification of this final detail to the reader.

Acknowledgment. This work was supported by the Charles Phelps Taft Memorial Fund and a Raybould Visiting Fellowship at the University of Queensland.

References

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3. R. S. Westfall, *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press, Cambridge, 1980.

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