$x < 1.169231$ she thought that she could live with the error in return for the ease in solving the problem.

I was defeated. Now I no longer use the sine when I teach calculus. If it works for the physicist it must work for the mathematician. I wonder, with what can I replace the cosine and tangent?

Determinantal Loci

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In a recent issue of this journal [**2**], an interesting problem is proposed: show that the matrix $xJ + A$ is singular for exactly one element $x \in K$, for no element $x \in K$, or for every element $x \in K$. The matrix A is n-square over a field K, and J is the n-square matrix of 1's. The matrix J is the dyad ee^T, where e is the $n \times 1$ matrix of 1's. Thus the problem suggests a generalization. Let

$$u_1 v_1^T + \cdots + u_p v_p^T \tag{1}$$

be a rank p matrix over K, written as a sum of rank 1 matrices $u_i v_i^T$, $i = 1, \ldots, p$, and let x_i, $i = 1, \ldots, p$, be indeterminates over the field K. Form the variable matrix

$$F(x_1, \ldots, x_p) = \sum_{t=1}^{p} x_t u_t v_t^T. \tag{2}$$

Then $F(x) + A$ is singular iff

$$\det(F(x) + A) = 0. \tag{3}$$

In (3), $F(x) = F(x_1, \ldots, x_p)$. The problem is to determine the variety in p-dimensional space K^p consisting of all specializations for which (3) holds. Since the matrix (1) is assumed to have rank p, it follows that $u_1, \ldots, u_p$ are linearly independent, as are $v_1, \ldots, v_p$. Hence, there exist non-singular matrices P and Q such that

$$Pu_t = Qv_t = e_t, \qquad t = 1, \ldots, p, \tag{4}$$

in which e_t is the column vector with 1 in position t and 0 elsewhere. Note that

$$PF(x)Q^T = \sum_{t=1}^{p} x_t e_t e_t^T$$
$$= D(x_1, \ldots, x_p)$$

in which $D(x) = D(x_1, \ldots, x_p)$ is the n-square diagonal matrix with $D(x)_{tt} = x_t$, $t = 1, \ldots, p$, and all other entries 0. If we set $B = PAQ^T$ the problem reduces to finding the locus of all points $x \in K^p$ for which the determinantal equation

$$\det(D(x) + B) = 0 \tag{5}$$

holds. To analyze the equation (5) we will use a formula for the determinant of a sum of two matrices that appeared in another recent issue of this journal [**1**]:

$$\det(D(x)+B)=\sum_{r}\sum_{\alpha,\beta}(-1)^{s(\alpha)+s(\beta)}\det(D(x)[\alpha|\beta])\det(B(\alpha|\beta)). \quad (6)$$

In the formula (6) the outer sum on r is over the integers $0,\dots,n$; for a particular r, the inner sum is over all strictly increasing integer sequences α and β of length r chosen from $1,\dots,n$; $D(x)[\alpha|\beta]$ is the r-square submatrix of $D(x)$ lying in rows α and columns β; $B(\alpha|\beta)$ is the $(n-r)$-square submatrix of B lying in rows complementary to α and columns complementary to β; and finally, $s(\alpha)$ is the sum of the integers in α. When $r=0$ the summand is taken to mean $\det(B)$ and when $r=n$ it is $\det(D(x))$. Since $D(x)$ is a diagonal matrix, the only summands in (6) that survive are those for which $\alpha=\beta$, and moreover, these sequences need only be chosen from $1,\dots,p$. Thus, (6) becomes:

$$\det(D(x)+B)=\sum_{r=0}^{p}\sum_{\alpha\in Q_{r,p}}x_{\alpha_1}\cdots x_{\alpha_r}\det(B(\alpha|\alpha)). \quad (7)$$

The set $Q_{r,p}$ in (7) is the totality of strictly increasing sequences of integers of length r chosen from $1,\dots,p$.

The problem in [**2**] occurs for $p=1$, in which case (7) becomes:

$$\det(B)+x_1\det(B(1|1)). \quad (8)$$

It is obvious that (8) can be 0 for exactly one specialization of x_1 to an element of K, for no such specialization, or for every such specialization. This argument does not depend on the form of J, but only on the fact that it has rank one. The situation is considerably more interesting when $p=2$. At this point we will assume that K is the field of real numbers so that a familiar geometric interpretation can be given to the set of points x satisfying (3). We also assume that $n\geq 3$.

Theorem. *Assume that L and M are rank one matrices for which $L+M$ is of rank two. Then the set of points (x_1,x_2) in the cartesian plane for which*

$$A+x_1L+x_2M \quad (9)$$

is singular is an arbitrary locus of one of the following types, and these only:

(i) the empty set;
(ii) a straight line;
(iii) an equilateral hyperbola whose axes are parallel to the coordinate axes;
(iv) a pair of straight lines parallel to the coordinate axes.

Proof. Clearly the situation in the statement is the case $p=2$ in the preceding discussion. Thus, for $p=2$, (7) becomes

$$\det(B)+x_1\det(B(1|1))+x_2\det(B(2|2))+x_1x_2\det(B(1,2|1,2)). \quad (10)$$

If the coefficient of x_1x_2 in (10) is not 0 then by a translation of axes the linear terms in x_1 and x_2 can be eliminated so that (10) takes the form

$$ax_1x_2+b. \quad (11)$$

If $b\neq 0$, then (11) is 0 for precisely the alternative (iii). If $b=0$, then (11) is 0 for (iv). On the other hand if the coefficient of x_1x_2 in (10) is 0, then the resulting linear form in x_1 and x_2 will be 0 for precisely one of the alternatives (i) or (ii).

In order to conclude that an *arbitrary* locus of types (i)–(iv) is defined by (5) we must show that given any elements a, b, c, d in K there exists an n-square matrix B for which

$$\det(B(1,2|1,2)) = a, \tag{12}$$

$$\det(B(1|1)) = b, \tag{13}$$

$$\det(B(2|2)) = c, \tag{14}$$

$$\det(B) = d. \tag{15}$$

We may assume $n = 3$ in establishing this statement. For, if $n > 3$ simply define B to be the direct sum of I_{n-3} with an appropriate 3-square matrix satisfying (12)–(15).

Case 1. $a \neq 0$.

Then set

$$B = \begin{bmatrix} \frac{c}{a} & \frac{bc - ad}{a^2} & 0 \\ 1 & \frac{b}{a} & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Case 2. $a = 0,\ c \neq 0$.

Then set

$$B = \begin{bmatrix} 0 & 0 & -c \\ 1 & 1 + \frac{d}{c} & -b \\ 1 & 1 & 0 \end{bmatrix}.$$

Case 3. $a = 0,\ c = 0,\ b \neq 0$.

Then set

$$B = \begin{bmatrix} \frac{d}{b} & 0 & 0 \\ 1 & 1 & -b \\ 1 & 1 & 0 \end{bmatrix}.$$

Case 4. $a = 0,\ c = 0,\ b = 0$.

Then set

$$B = \begin{bmatrix} 0 & 1 & 1 \\ d & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Acknowledgment. This research was supported by a grant from the Lawrence Livermore Laboratories of the University of California.

References

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The Probability that $(a, b) = 1$

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Note (by Henry L. Alder): To ask what the probability is for two positive integers a and b to have the greatest common divisor 1 is a natural question and was raised by students in my beginning number theory class in the Fall quarter of 1989. I answered it and gave a traditional, rather lengthy proof calling on considerable prior knowledge of number theory. The above named two students (the first a 16-year old freshman, the second a 17-year old high school student) came up with the following much shorter proof. I encouraged them to share it with the readers of the *College Mathematical Journal* who might be asked the same question in their classes.

Let g be the greatest common divisor of two integers a and b, that is $g = (a, b)$ and let p be the probability* that $g = 1$. We will first show that the probability that $g = n$ for $n = 1, 2, \ldots$ is p/n^2.

Clearly the probability that n divides both a and b is $1/n^2$. The probability that no proper multiple of n divides both a and b is the same as the probability that $(a/n, b/n) = 1$, which is p. Thus, the probability that $g = n$ is p/n^2.

The sum of the probabilities that $g = n$ for $n = 1, 2, \ldots$ must equal 1, so that

$$\sum_{n=1}^{\infty} \frac{p}{n^2} = 1.$$

Solving for p, we obtain

$$p = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2}.$$

*The probability refers, of course, to the

$$\lim_{N \to \infty} \frac{\#\{(a,b):(a,b) = 1, 1 \le a \le N, 1 \le b \le N\}}{\#\{(a,b): 1 \le a \le N, 1 \le b \le N\}}.$$

That this limit exists is well known. (See, for example, A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Vol. I, Holden-Day, San Francisco, 1964, pp. 202–4).