Investigating Possible Boundaries Between Convergence and Divergence

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In the following, we describe a classroom discussion that can be used to supplement the material on the integral test for convergence and divergence of series.

One of the first applications of the integral test is to show that the series $\sum 1/n^p$ converges for $p > 1$ and diverges for $p \leq 1$. [Throughout this note, we assume that p is a positive real number, and we omit the indices of infinite series because we are concerned only with their tails.] Recently one of our students remarked that the harmonic series $\sum 1/n$ acts like a boundary between convergence and divergence. Since multiplying n in the denominator by a positive power of n (no matter how small) produced a convergent series, this student conjectured that any series whose denominator involves n multiplied by an expression that is always greater than 1 must be convergent. In other words, any series of positive terms which are term by term less than $1/n$ must converge. After it was pointed out that multiplication by a constant does not change convergence or divergence, this conjecture was quickly modified to what amounted to the following:

$$\text{If } a_n \to \infty \text{ as } n \to \infty, \text{ then } \sum \frac{1}{a_n \cdot n} \text{ converges.} \tag{1}$$

Since the unit considered in class involved the integral test, it was natural to investigate this conjecture by considering the case where $a_n = g(n)$ for some continuous function g. After a few false starts, the function $g(x) = \ln(x)$ was shown to produce a counterexample to (1). In fact, $\sum 1/n(\ln(n))^p$ diverges for all $p \leq 1$. Since this series converges for $p > 1$, it was natural to consider whether n can be replaced by $n\ln(n)$ in (1).

This new conjecture fails as can be seen by letting $a_n = \ln(\ln(n))$ and applying the integral test to $\sum 1/n \cdot \ln(n) \cdot \ln(\ln(n))$. This pattern can be continued indefinitely. Let

$$f^0(x) = x, \quad f^1(x) = \ln(x) = f^0(\ln(x)), \quad f^2(x) = \ln(\ln(x)) = f^1(\ln(x)),$$
$$\text{and } f^{k+1}(x) = f^k(\ln(x))\ k \geq 1.$$

Since the derivative of $f^k(x)$ equals $1/f^0(x) \cdot f^1(x) \cdots f^{k-1}(x)$, it follows by the integral test that $\sum 1/f^0(n) \cdot f^1(n) \cdots f^{k-1}(n) \cdot (f^k(n))^p$ converges for $p > 1$ and diverges for $p \leq 1$. Once again, $p = 1$ seems to be a dividing value between convergence and divergence. However, (1) still fails to be true when n is replaced by $f^0(n) \cdot f^1(n) \cdots f^k(n)$, since $a_n = f^{k+1}(n)$ gives the divergent series

$$\sum \frac{1}{f^{k+1}(n) \cdot f^0(n) \cdot f^1(n) \cdots f^k(n)} = \sum \frac{1}{f^{k+1}(n)} Df^{k+1}(n).$$

Thus, the divergent series $\sum 1/f^0(n) \cdot f^1(n) \cdots f^{k+1}(n)$ slips between the divergent series $\sum 1/f^0(n) \cdot f^1(n) \cdots f^k(n)$ and the convergent series $\sum 1/f^0(n) \cdot f^1(n) \cdots f^{k-1}(n) \cdot [f^k(n)]^p$ for $p > 1$.

The above example shows that we cannot use this approach to produce a boundary between divergence and convergence. The question arises: Will *any* procedure produce a "smallest" divergent series? The answer to this question is no, as can be seen from the following theorem in [**1**].

Theorem. *If $\sum a_n$ is a divergent series of positive reals, then there exists a sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers that converges to zero, but for which $\sum \epsilon_n \cdot a_n$ still diverges.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. We first show $\sum_{k=1}^{\infty}(s_{k+1} - s_k)/s_{k+1}$ diverges. For any $m \in \mathbb{N}$, choose $n \in \mathbb{N}$ such that $s_{n+1} > 2s_m$. Since $\{s_k\}_k^{\infty}$ is non-decreasing,

$$\begin{aligned}\sum_{k=m}^{n} \frac{s_{k+1} - s_k}{s_{k+1}} &\geq \sum_{k=m}^{n} \frac{s_{k+1} - s_k}{s_{n+1}} \\ &= \frac{1}{s_{n+1}}[(s_{m+1} - s_m) + (s_{m+2} - s_{m+1}) + \cdots + (s_{n+1} - s_n)] \\ &= \frac{s_{n+1} - s_m}{s_{n+1}} \\ &> \frac{s_{n+1} - \frac{1}{2}s_{n+1}}{s_{n+1}} = \frac{1}{2}.\end{aligned}$$

Thus, the partial sums of the series $\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}}$ do not form a Cauchy sequence, and so $\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}} = \infty$. Since $s_{k+1} - s_k = a_{k+1}$,

$$\sum_{k=1}^{\infty} \frac{s_{k+1} - s_k}{s_{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{s_k}.$$

Now let $\epsilon_k = 1/s_k$. Then $\epsilon_k \to 0$ and $\sum_{k=2}^{\infty} \epsilon_k a_k = \infty$. ∎

It is worthwhile to make students realize that there is no specific series that can be used to establish a boundary between the set of all divergent positive-termed series and the set of all convergent series. Readers interested in a more detailed historical development of this topic may wish to consult [**2**].

References

1. R. R. Goldberg, *Methods of Real Analysis*, 2nd. ed., John Wiley & Sons, 1976, 71–72.
2. K. Knopp, *Theory and Application of Infinite Series*, Hafner, 1947, 290–291.

Two Quick Combinatorial Proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.

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A standard exercise in mathematical induction in many discrete mathematics classes is to prove the identity $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$. Alternative proofs are possible that allow this identity to be appreciated from different perspectives. For instance, in [**2**] seven different geometric proofs are presented.