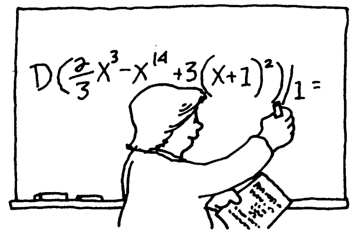
CLASSROOM CAPSULES

EDITOR

Frank Flanigan
Department of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

Single Equations Can Draw Pictures

Keith M. Kendig, Cleveland State University, Cleveland, OH 44115

Analytic geometry is more powerful than Euclidean geometry, but students find that analytic geometry leaves many of the familiar objects of Euclidean geometry behind. Thus, any ellipse has a polynomial defining it, but where's the polynomial defining a triangle or a square or two intersecting circles? The purpose of this paper is to bring the two worlds of Euclidean and analytic geometry a little closer by one of the themes of higher mathematics—the versatility of polynomials.

In high school I'd just learned the equation of a circle. During a school assembly a friend and I tried unsuccessfully to find a single equation that would give two concentric circles. The next day my friend appeared in math class like one bearing a rare treasure. He'd asked his father who, conveniently, was a mathematics professor. On a piece of paper there was neatly written:

The equation of two circles of radii 1 and 2, centered at the origin, is

$$x^4 + 2x^2y^2 + y^4 - 5x^2 - 5y^2 = -4.$$

We showed it to our math teacher; he and our class spent a few minutes substituting in various points. It was like magic—the incredible equation always worked.

Years later, the big light turned on. The secret to that incredible equation turned out to be disarmingly simple: write the equations of the two separate circles so 0 is on the right of the equals sign:

$$x^2 + y^2 - 1 = 0 \quad \text{and} \quad x^2 + y^2 - 4 = 0.$$

Then just multiply the equations together!

Why does this work? Any point on the smaller circle makes $x^2 + y^2 - 1$ zero, so it certainly does the same to $(x^2 + y^2 - 1)(x^2 + y^2 - 4)$. Likewise for any point on the other circle. And if a point isn't on either circle, then neither $x^2 + y^2 - 1$ nor

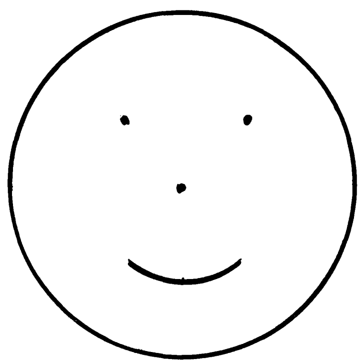
$x^2 + y^2 - 4$ is zero, so the product isn't either. Thus the equation describes exactly the two circles.

This principle gives students a wonderful toolbox for generating equations for all sorts of simple pictures. Circles, lines, parabolas, ellipses, hyperbolas—write their equations with zero on the right-hand side, multiply them together, and you have the equation for the union of their figures. (Cf. [**3**], Theorem 3.13.) You can even obtain the point (a, b) by looking at it as a circle of zero radius, $(x-a)^2 + (y-b)^2 = 0$.

Sometimes it is difficult to sketch a contour curve $p(x, y) = 0$ by hand, but mathematical graphics programs such as *Mathematica*, *Macsyma* and *Matlab* have made this much easier. However, if $p(x, y)$ is too complicated, even these programs may miss some of the fine details of the curves.

Line segments: Plain and fancy. A single equation for a line segment would certainly enrich the toolbox quite a bit. If we restrict ourselves to polynomial equations $p(x, y) = 0$, then there isn't one, but we can approximate a line one unit long by using an ellipse with major axis of length 1 and width less than the thickness of the pencil line.

We can now almost (but not quite) draw a "happy face":



The large circle, and the three small circles for the eyes and nose, are straightforward. But how about the smile? We start with a "line segment" of slope 1—that is, a very thin ellipse whose major axis has slope 1. One such ellipse is

$$x^2 - (2-\varepsilon)xy + y^2 = \frac{\varepsilon}{4} \tag{1}$$

($\varepsilon > 0$, and small—.001 will do nicely). It looks like the line segment from $(-\frac{1}{2}, -\frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$. It is instructive to use a computer graphics package to discover how a change in ε affects the graph of (1). Note that if we replace x by x^2 in the equation $y = x$, we form a parabola. If we similarly replace each x by x^2 in (1), the segment (ellipse) will turn up smiling. We'll replace x by $2x^2$, so it will smile even more. The equation of this smile is

$$4x^4 - 2(2-\varepsilon)x^2y + y^2 = \frac{\varepsilon}{4}.$$

Then the following equation produces a "happy face" of radius 2, with its nose at the origin:

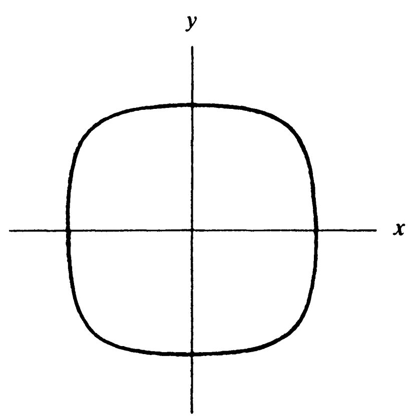
$$(x^2+y^2-4)\cdot(x^2+y^2)\cdot((x-1)^2+(y-1)^2)\cdot((x+1)^2+(y-1)^2)$$

$$\cdot\left(4x^4-2(2-\varepsilon)x^2y+y^2-\frac{\varepsilon}{4}\right)=0.$$

Of course one could multiply this out to conceal all tracks and amaze the unsuspecting.

The trick of making the line segment smile can be generalized to make the segment take the shape of the graph of any polynomial $p(x)$ by replacing every occurrence of x in (1) with $p(x)$. Thus the happy face can be given a quizzical look by replacing x with x^3 in equation (1). Just as replacing x by $p(x)$ makes the line segment follow the graph of $y=p(x)$, replacing y by $q(y)$ makes it follow the graph of $x=q(y)$. Finally, we may make the mouth open (keeping roughly the same expression), by making ε in the ellipse's equation larger, for then the ellipse isn't so thin.

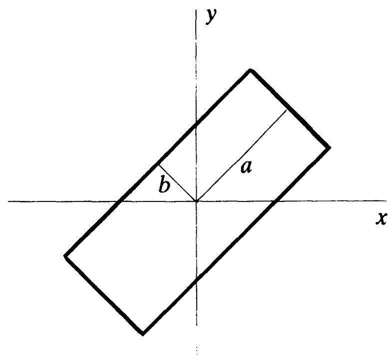
A new angle on corners. By using various "line segments," one can construct equations for all sorts of polygons. However, for many of these, there is an easier way. For example, $x^4+y^4=1$ gives this:



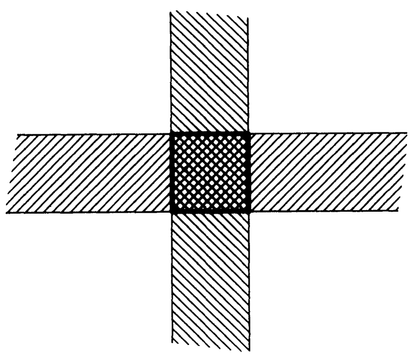
As m gets large, the corners of $x^{2m}+y^{2m}=1$ get sharper and the figure will eventually get inside any arbitrarily accurate pencil sketch of a square. Likewise, a near rectangle of dimensions $2a$ by $2b$ can be approximated by $(x/a)^{2m}+(y/b)^{2m}=1$, m large. And any of these rectangles can be rotated through an angle θ by replacing x by $x\cos\theta+y\sin\theta$, and y by $x\cos\theta-y\sin\theta$. Thus, for large m, the locus of

$$\left(\frac{x+y}{\sqrt{2}a}\right)^{2m}+\left(\frac{x-y}{\sqrt{2}b}\right)^{2m}=1$$

looks like this:



Why does raising to large even powers create squares and rectangles? Observe that for any positive integer m, $x^{2m} = 1$ describes the two vertical lines $x = 1$ and $x = -1$. In the region between these lines, x^{2m} is positive and less than 1, and if m is very large, x^{2m} is nearly zero there except very close to either line. Also, outside the strip, x^{2m} quickly gets very large. Likewise, $y^{2m} = 1$ describes two horizontal lines, one unit above and one unit below the x-axis. In order for the sum of x^{2m} and y^{2m} to equal 1 when m is very large, (x, y) must be close to the boundary of the intersection of the two strips:



One can create strips having any position or width:

$$\left(\frac{ax + by + c}{d}\right)^{2m} = 1$$

defines two lines equidistant from the line $ax + by + c = 0$. By appropriately choosing d, one can make the strip any desired width. Now there's no reason why we must limit ourselves to only two strips! The intersection of strips of various widths and slopes, can give any convex polygonal region. For example, for m sufficiently large,

$$x^{2m} + y^{2m} + \frac{(x+y)^{2m}}{2^m} + \frac{(x-y)^{2m}}{2^m} = 1$$

stays as close to a regular octagon as one wants.

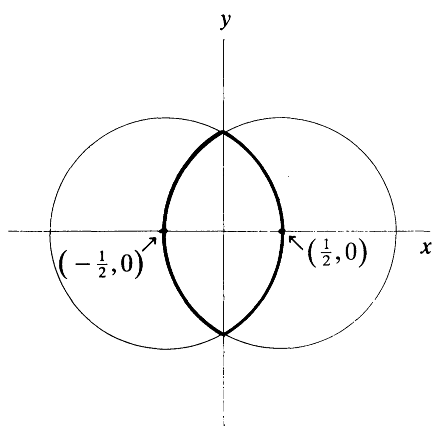
Is there a connection between our construction of smiling faces and rectangles? The smiling face is a union of simple curves which we found by multiplying together the polynomials of the component "building blocks" and setting the result equal to zero. *Multiplying* the curves $p(x, y) = 0$ produced the *union* of the curves. For nonnegative polynomials $q(x, y)$, *adding* the $[q(x, y)]^m$ and setting this sum equal to 1 produces a curve related to *intersection*. This curve lies near the boundary of the intersection of the regions where the $q(x, y)$ take on values less than 1. Thus, for example,

$$\left(x+\tfrac{1}{2}\right)^2+y^2=1 \quad \text{and} \quad \left(x-\tfrac{1}{2}\right)^2+y^2=1$$

give two unit circles centered at $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$. For large m,

$$\left(\left(x+\tfrac{1}{2}\right)^2+y^2\right)^m+\left(\left(x-\tfrac{1}{2}\right)^2+y^2\right)^m=1$$

yields the heavily drawn football-shaped curve:



Everything we've done generalizes to n dimensions. For instance in three dimensions one can add high powers of nonnegative polynomials to create shapes with vertices and edges, and one can multiply polynomials to get the union of spheres, points, boxes, polyhedra, and so forth. A very thin pancake-shaped ellipsoid serves nicely as a disk. If its two longer axes lie in the plane $z = x + y$, then replacing x by x^2 and y by y^2 creates a bowl that will catch rainwater. Replacing x by $-x^2$ and y by $-y^2$ turns the bowl upside down. Finally, replacing x by x^2 and y by $-y^2$ produces a disk warped into a saddle shape.

By using appropriate elementary building blocks (simple polynomials) and putting them together with multiplication (producing unions), addition (making corners), or composition (introducing warping), one can create a vast array of interesting loci. There's almost no limit to the figures one can craft, using garden-variety polynomials and a bit of imagination!

The interested reader will find further general information about curves defined by polynomials in [**1**]; a shorter account with a broader perspective is contained in [**2**].

References

1. K. Kendig, *Elementary Algebraic Geometry*, Springer-Verlag, New York, 1977, Chapter II.
2. ______, Algebra, Geometry, and algebraic geometry: Some interconnections, *American Mathematical Monthly* 90 (1983) 161–173.
3. A. Seidenberg, *Elements of the Theory of Algebraic Curves*, Addison-Wesley, Reading, MA, 1968.

The Snowplow Problem Revisited

Xiao-peng Xu, University of Massachusetts, Amherst, 01003

A classic problem in elementary differential equations, commonly attributed to R. P. Agnew [*Differential Equations*, McGraw-Hill, 1942, pp. 30–32], is the following:

One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?

The problem is usually solved by setting $t = 0$ at noon, setting up the relevant differential equation, finding its general solution and then using the conditions of the problem to eliminate all the arbitrary constants. This procedure involves a fair amount of algebra which, if not done carefully, can be quite tedious. There is, however, a quick and easy way that avoids most of this algebra. It goes as follows:

Let t denote time, measured in hours, and let $t = 0$ at 1:00 P.M. Let t_0 be the time it started snowing. Let $y(t)$ denote the distance traveled by the snowplow, measured in miles. Let $h(t)$ be the height of the snow at time t, so that $h(t_0) = 0$. Let s denote the rate of the snowfall, measured in any suitable units. Then, since it was snowing at a steady rate, $h(t) = s(t - t_0)$. Assume that the width of the snowplow is one unit and let k be the amount of snow that the plow can remove per unit time. Then we have

$$h(t)\frac{dy}{dt} = k \quad \text{or} \quad \frac{dy}{dt} = \frac{c}{t - t_0},$$

where $c = k/s$. Now (and this is the trick) instead of finding the general solution of this differential equation, we note that

$$2 = c\ln(t - t_0)\Big|_{-1}^{0} \quad \text{and} \quad 1 = c\ln(t - t_0)\Big|_{0}^{1}$$

whence $2\ln[(t_0 - 1)/t_0] = \ln[t_0/(t_0 + 1)]$, and a very little algebra yields $t_0^2 + t_0 - 1 = 0$, so that $t_0 = (-1 - \sqrt{5})/2$.

The Differentiability of Sin *x*

David A. Rose, East Central University, Ada, OK 74820

That $\sin x$ is differentiable with derivative $\cos x$ implies that $\sin x$ has derivative 1 at $x = 0$, i.e.,

$$\operatorname*{limit}_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$