problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. [**2**]

References

1. G. Polya, *How To Solve It*, Princeton University Press, Princeton, NJ, 1946, p. 14.
2. L. A. Steen and D. J. Albers, eds., *Teaching Teachers*, *Teaching Students*, Birkhauser, Boston, 1981, p. 89.

Least Squares and Quadric Surfaces

Donald Teets, South Dakota School of Mines and Technology, Rapid City, SD 57701-3995

We have all seen fairly difficult problems in college algebra texts that would be easy "if only the student knew calculus." But have you ever seen a difficult problem in a calculus text that would be easier if only the student *didn't* use calculus? The purpose of this note is to describe such a problem.

The derivation of the least squares regression line $f(x) = ax + b$ for the n data points $(x_1, y_1), \ldots, (x_n, y_n)$ (where $n \geq 2$ and the x_i's are not all the same) is commonly presented as an application of minimizing a multivariable function [**1**], [**2**], [**3**]. The standard approach to this problem is to minimize the sum of the squared errors

$$s(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \tag{1}$$

by setting the partial derivatives $s_a(a, b)$ and $s_b(a, b)$ equal to zero and solving for a and b, obtaining

$$a = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}, \qquad b = \frac{1}{n}\left(\sum y_i - a\sum x_i\right). \tag{2}$$

(Here and in the remainder of this note, Σ means $\Sigma_{i=1}^{n}$.) This, of course, is insufficient to show that these values minimize $s(a, b)$; thus we find the following exercise in [**3**]: "Use the Second-Partials Test to verify that the formulas given for a and b yield a minimum." (The reader is invited to try this exercise before reading further!)

For the function $s(a, b)$ to have a minimum at the point (a, b) given in (2), the second-partial test requires that

$$s_{aa}(a, b)s_{bb}(a, b) - [s_{ab}(a, b)]^2 > 0$$

at that point. Upon computing the derivatives, this reduces to

$$n\sum x_i^2 - \left(\sum x_i\right)^2 > 0. \tag{3}$$

Now (3) can be established by a rather tricky induction on n, or by applying the Cauchy-Schwarz inequality to the vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (1, \dots, 1)$, or by other clever algebraic manipulations (e.g., [**1**] p. 1031). Unfortunately, even if a few students are able to follow one of these arguments, the second-partials test is a test for *relative* extrema, and the least squares problem demands an *absolute* minimum of $s(a,b)$! Faced with making an already difficult calculus problem even harder, we choose an entirely different approach.

An understanding of the geometry of this problem provides the key to a solution involving no calculus. One may quickly recognize that the graph in three-dimensional *abs*-space of (1) is a quadric surface, though the term $2ab\Sigma x_i$ which occurs in the expansion of $s(a,b)$ obscures the nature of this surface. To eliminate this term, we let $X = x - \bar{x}$ (where $\bar{x} = (1/n)\Sigma x_i$) and $X_i = x_i - \bar{x}$. Note crucially that $\Sigma X_i = 0$. The least squares regression line is now

$$f(x) = ax + b = a(X + \bar{x}) + b = aX + c$$

where $c = a\bar{x} + b$, and we seek the absolute minimum of

$$s(a,b) = \sum (ax_i + b - y_i)^2 = \sum (aX_i + c - y_i)^2,$$

which we call $S(a,c)$.

Geometry and basic algebra can now make clear what calculus could not. Expanding $S(a,c)$ we find that the mixed term $2ac\Sigma X_i$ vanishes, and so we may complete the square separately in a and c, obtaining

$$S(a,c) = \left(\sum X_i^2\right)\left[a - \frac{\Sigma X_i y_i}{\Sigma X_i^2}\right]^2 + n\left[c - \frac{\Sigma y_i}{n}\right]^2$$
$$+ \sum y_i^2 - \frac{(\Sigma X_i y_i)^2}{\Sigma X_i^2} - \frac{(\Sigma y_i)^2}{n}.$$

The graph in *acS*-space of this function is an elliptic paraboloid opening upward, and it is now obvious, both analytically and geometrically, that the absolute minimum of $s(a,b) = S(a,c)$ occurs at

$$a = \frac{\Sigma X_i y_i}{\Sigma X_i^2}, \qquad c = \frac{\Sigma y_i}{n}, \qquad b = c - a\bar{x}$$

as given in (2). Moreover, the value of s at this point can now be determined at a glance.

For this and other applications in this spirit, see [**4**].

References

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3. R. Larson and R. Hostetler, *Calculus with Analytic Geometry*, alternate 3rd edition, D. C. Heath, Lexington, MA, 1986.
4. I. Niven, *Maxima and Minima without Calculus*, Mathematical Association of America, Washington DC, 1981.$h/r \quad V^{1/3}/K$