Divide the numerator and the denominator of the fractions by h and let $h \to 0$ to obtain (if $f''(a) \neq 0$)

$$x_0 = \lim_{h \to 0} x_h = \frac{af''(a) - f'(a)^3 - f'(a)}{f''(a)} = a - \frac{f'(a)^3}{f''(a)} - \frac{f'(a)}{f''(a)}$$

and

$$y_0 = \lim_{h \to 0} y_h = \frac{1 + f'(a)^2 + f(a)f''(a)}{f''(a)} = f(a) + \frac{f'(a)^2}{f''(a)} + \frac{1}{f''(a)}.$$

So if $P(a, f(a))$ is such that $f''(a) \neq 0$ then

$$R = \left(a - \frac{f'(a)^3}{f''(a)} - \frac{f'(a)}{f''(a)}, f(a) + \frac{f'(a)^2}{f''(a)} + \frac{1}{f''(a)} \right).$$

The distance from R to the point $P(a, f(a))$ is

$$\frac{\left(1 + f'(a)^2\right)^{3/2}}{|f''(a)|},$$

which is the radius of curvature of C at $P(a, f(a))$. Moreover R is the center of the circle of curvature of C at $P(a, f(a))$, that circle that best approximates C at $P(a, f(a))$.

The above is undoubtedly known to differential geometers, but it is surprising that it has not made its way into calculus textbooks. Using a specific curve and a specific fixed point on the curve, it would be an instructive exercise to have students compute R by a limiting process. Indeed, when experienced calculus teachers were asked what happens to the point $R(h)$ as $h \to 0$, many felt that this intersection point would go to infinity.

This is also a nice introduction to the concept of curvature of a planar curve at a point. There are two extremes, a line and a circle. For a line, all normals are parallel and R is "infinity". For a circle, all normals pass through the center of the circle and R is the center of the circle. For a general curve $y = f(x)$, the analysis above leads one to observe that if $f''(a) \neq 0$, the curve acts locally like a circle because R is finite. If $f''(a) = 0$, then the curve acts locally like a line.

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A Picture for Real Arithmetic

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By means of a stereographic projection, one can map the real line to a circle and thus have a picture which stays on the page, instead of running off it. By means of geometric procedures, the product and sum of two numbers can then be constructed. Various properties of arithmetic, such as the product of two negatives being positive, can then be visually presented.

For a concrete example, consider an (x, y)-coordinate system, and map the x-axis to the unit circle with $(0, 1)$ as the projection point (Figure 1).

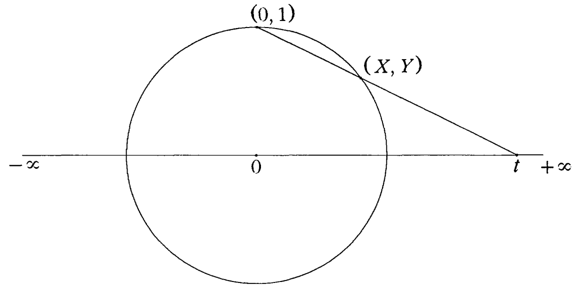

Figure 1

Each point (X, Y) on the circle can thus be labeled with a real number t, except the point $(0, 1)$ which is labeled $\pm\infty$, i.e. $+\infty$ and $-\infty$ are added to the real line and identified, as is done for the real projective plane. Explicitly, the mapping is

$$X(t) = \frac{2t}{t^2+1}, \quad Y(t) = \frac{t^2-1}{t^2+1}, \quad \text{and conversely} \quad t(X, Y) = \frac{X}{1-Y} = \frac{1+Y}{X}.$$

If either of the last two expressions is indeterminate, i.e. $\frac{0}{0}$, the other one gives the specific answer. Note the symmetries, $X(-t) = -X(t)$, $Y(-t) = Y(t)$ and $X(1/t) = X(t)$, $Y(1/t) = -Y(t)$, exhibited in Figure 2 for a few values of t.

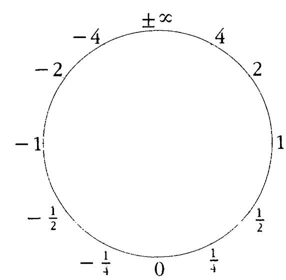

Figure 2

To picture multiplication and addition, the lines $x = 0$ and $y = 1$, each with a point $\pm\infty$ added, will be used as calculation lines respectively:

Multiplication Rule. *Draw the line through numbers a and b on the circle. It will intersect the $x = 0$ line at some point (possibly at $\pm\infty$). Draw the line through this point and the number 1 on the circle. It intersects the circle at number ab.* (Figure 3)

Addition Rule. *The line through a and b intersects $y = 1$ at some point. Draw the line through this point and 0. It intersects the circle at $a + b$.* (Figure 4)

When a and b coincide, one interprets "the line through them" as the tangent to the circle at that point.

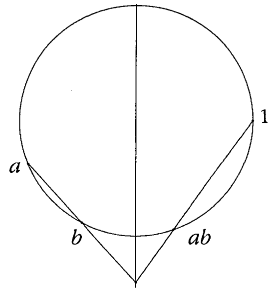

Figure 3

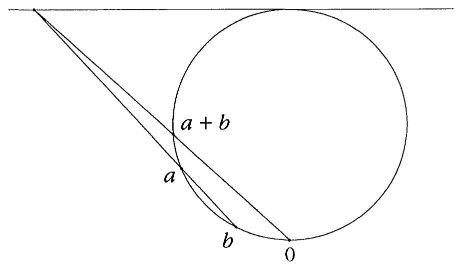

Figure 4

The proof that these rules actually give ab and $a + b$ are left as exercises. (Hint: Show that the line through a and b intersects the $x = 0$ line at $y = \frac{ab+1}{ab-1}$. If one then replaces a by 1 and b by ab, it follows that the line through 1 and ab intersects the $x = 0$ line at the same point. Similarly, show that the line through a and b intersects the $y = 1$ line at $\frac{2}{a+b}$.)

From the multiplication rule, clearly $1a = a$, $0a = 0$ for $a \neq \infty$, and $\infty a = \infty$ for $a \neq 0$. The line through a and $1/a$, for $a \neq \infty$, is parallel to the calculation line, intersecting it at the $\pm\infty$ point, and the line through it and 1 is the tangent at 1, so that $a(1/a) = 1$. The line through two negative numbers always intersects the calculation line outside the circle, providing a picture of why their product is a positive number. Similarly, the intersection for a positive and a negative number is inside the circle showing that their product is negative. Of particular interest is the line through 0 and ∞, which coincides with the calculation line and thus "intersects" at every point, so $0 \cdot \infty$ is all possible values. This provides a picture of just what we mean when we say that $0 \cdot \infty$ is indeterminate. To picture this more explicitly, let the line through a and b intersect the calculation line at some arbitrary point $y = c$ and then rotate the former line about this point until it coincides in a limiting process with the latter. Then $ab \to c$ in the limit as $a \to 0$ and $b \to \pm\infty$, or vice versa.

From the addition rule, clearly $0 + a = a$, and $\infty + a = \infty$ for $a \neq \infty$. The line through a and $-a$ is parallel to the calculation line, so $a + (-a) = 0$ for $a \neq \infty$. The line for $(\pm\infty) + (\pm\infty)$ coincides with the calculation line so the sum is indeterminate. That's because the limit $a \to \pm\infty$ can go either way around the circle. The usual interpretation that $\infty - \infty$ is indeterminate and $\infty + \infty = \infty$ occurs when $+\infty$ and $-\infty$ are not identified. These two results can be pictured explicitly on the circle by letting $a \to +\infty$ mean counterclockwise motion on the circle and $a \to -\infty$ mean clockwise motion.

Some Generalizing Remarks. 1) If one multiplies every number on the circle by a nonzero real number, it's easy to see that the calculation rules still hold. This means the real line can be mapped to the circle such that 1 can be any point on the circle other than 0 and ∞. More difficult to see is that one can add a real number to any number on the circle, and, with the calculation line for multiplication being the line through 0 and ∞, the calculation rules still hold. Thus 0 can be any point on the

circle other than ∞. To show these two results formally, stretch and translate the original x-axis by using the parameter t', where $t = ct' + d$, so 0 now occurs where d was and 1 where $c + d$ was. The points on the circle are similarly relabeled and the line now through 0 and ∞ on the circle (the multiplication calculation line) is $x = d(1 - y)$. This line is then intersected by the line through a and b on the circle (as relabeled) at $y = \frac{c^2ab - d^2 - 1}{c^2ab - d^2 + 1}$. For addition the calculation line has not changed (the tangent to ∞) and the line through a and b intersects it at $x = \frac{2}{c(a+b) + 2d}$.
2) The picture provided above, using the circle with a calculation line and an identity point to define an operation, is a specific case of the following very general result.

In the real projective plane, given a conic, a line (called the calculation line), and a point (called the identity) which is on the conic and not on the line, the points on the conic which don't intersect the line form a group with respect to the operation defined by the calculation rule.

A proof of this is given in **[4]**, where associativity of the operation follows from Pascal's Theorem. For the circle we've seen what this group is when the line intersects it in two distinct points and when it does so in one point. Further investigation reveals this group is complex number multiplication on the unit circle when the line intersects it in no points. (Hint: These numbers have the same position on the unit circle as they have in the complex plane when the calculation line is the line at infinity in the projective plane.) Because there is a projectivity which maps a circle in the real projective plane to an arbitrary conic leaving lines, intersections and tangencies invariant, the group in the very general result above is isomorphic to

1) nonzero real number multiplication when the line intersects the conic in two distinct points,
2) real number addition when the line intersects the conic in one point,
3) complex number multiplication on the unit circle when the line intersects the conic in no points.

Besides the possibility of constructing other interesting "pictures" of these groups on parabolas and hyperbolas, there are other investigations to pursue such as the transition from one group to the next as the calculation line is moved by a continuous parameter.

Historical Note. The idea of doing arithmetic geometrically goes back a long way. For example, one reads in [**1**, p. 58], "During the Hellenistic period... even the arithmetic of whole numbers was constructed geometrically." In the middle of the 19th century an arithmetic using points on a line was constructed using projective geometry, which means using only a straightedge. Sources [**2**, p. 155 and **6**, p. 141] both cite von Staudt **[5]** as presenting an arithmetic of equivalence classes consisting of sets of 4 points on the line. They further cite Hessenberg **[3]** as simplifying this to an arithmetic of simply points on the line. Further simplification follows upon using a projectivity to map the line to a conic, which results in the calculation rule given above for a conic [**2**, pp. 155–160 and **6**, pp. 231–232]. Source **[2]** includes a disk, described in Appendix 2, which has on it "The Arithmetic of Points on a Circle".

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Integrals of $\cos^{2n}x$ and $\sin^{2n}x$

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Integrals of the even powers of sine and cosine are notoriously difficult, and most texts approach them either by half-angle identities for $\cos^2 x$ and $\sin^2 x$ or by using reduction formulas. There is also a nice application of complex numbers that allows closed formulas to be derived quite easily. It is based on DeMoivre's formula

$$z^n = r^n(\cos n\theta + i \sin n\theta). \tag{1}$$

Let

$$z = \cos x + i \sin x,$$

so

$$\frac{1}{z} = \cos x - i \sin x$$

and, therefore,

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right). \tag{2}$$

Applying the binomial formula to (2) gives

$$\cos^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k} \tag{3}$$

and

$$\sin^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{n-k} z^{2n-2k}. \tag{4}$$

From DeMoivre's formula (1) it follows that

$$z^{2n-2k} = \cos(2n-2k)x + i \sin(2n-2k)x,$$

which transforms (3) and (4) into

$$\cos^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} \cos(2n-2k)x + \frac{i}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} \sin(2n-2k)x$$