Figure 3 is also the scene of an interesting fact about the relationship between angles $\angle$TPF and $\angle$TPQ. Since $PF/PQ = e$, we can divide numerator and denominator by PT and conclude that $(\cos\angle TPF)/(\cos\angle TPQ) = e$.

References

1. Allan B. Cruse and Millianne Granberg, *Lectures on Freshman Calculus*, Addison-Wesley, Reading, MA, 1971, §I.2.
2. B. A. Troesch, Archimedes' method for the reflections on the ellipse, *Mathematics Magazine* 64 (1991) 262–263.
3. Zalman P. Usiskin, A pretrigonometry proof of the reflection property of the ellipse, *College Mathematics Journal* 17 (1986) 418.

Lottery Drawings Often Have Consecutive Numbers

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There are few random processes more avidly watched than the state lottery drawings in which six numbered balls are chosen from a set of 44 (in Louisiana; other states vary.) People have been surprised to notice that the winning selection often contains two consecutive numbers. We can compute the probability of this happening, and see that it is actually greater than one half.

The probability can be computed using standard counting techniques found in any advanced combinatorics book; in fact, it appears implicitly as a problem in [**1**, p. 72] and [**4**, p. 59]. The purpose of this note is to make the solution accessible to the student who knows only the elementary principles of counting.

Let us begin by counting the number of ways that q identical objects can be distributed among p labelled boxes. Think of the objects lined up in a row. We can assign them to the boxes by inserting $p-1$ markers into the row: Box 1 will get the objects (if any) to the left of marker 1; box 2 will get those between marker 1 and marker 2; etc.

How many ways can this be done? What we have is a row of $q+p-1$ things: q objects and $p-1$ markers. The question is how many ways we can choose which $p-1$ of the things will be the markers: $C(q+p-1, p-1)$. We sum this up as:

Theorem. [**1**], [**4**] *The number of ways that q identical objects can be distributed among p labelled boxes is* $C(q+p-1, p-1)$.

This theorem is simple enough, but as in so many counting problems, the hard part is deciding for our problem what we should consider to be the objects and the boxes.

Consider the general case of choosing k numbered balls from a set of n. Think of the $n-k$ losing balls as objects and the k winning balls as dividing them into $k+1$ boxes: those before the first, those between the first and second, etc.

Using the theorem, the number of ways this can be done is

$$C(n-k+[k+1]-1, [k+1]-1) = C(n,k)$$

as we knew it must be.

Now, possibly some of these boxes will be empty; in fact, a box will be empty precisely when two winning numbers are consecutive. So to count the number of ways that no pair of winning numbers will be consecutive, we should count the number of ways to put the $n - k$ losing balls into the $k + 1$ boxes so that none of the boxes (except possibly for the first and the last) will be empty.

But it is easy to handle the constraint that a given box must be non-empty: Begin by putting a ball in it before distributing the other balls. In our case we put a ball in each of the $k - 1$ boxes (other than the first and last) and then distribute the remaining $(n - k) - (k - 1) = n - 2k + 1$ balls at will among the full set of $k + 1$ boxes. By the theorem, the number of ways that this can be done is

$$C(n - 2k + 1 + [k + 1] - 1, [k + 1] - 1) = C(n - k + 1, k).$$

Thus the probability that k numbered balls chosen from a set of n will have no two consecutive is:

$$C(n - k + 1, k)/C(n, k).$$

When $n = 44$ and $k = 6$ this is 0.462. The probability that there will be some two consecutive is therefore $1 - 0.462 = 0.538$.

Figure 1 gives the probabilities that 6 balls drawn from n will have some two consecutive, where n ranges from 10 to 100. When n is 48 the probability is just over one half.

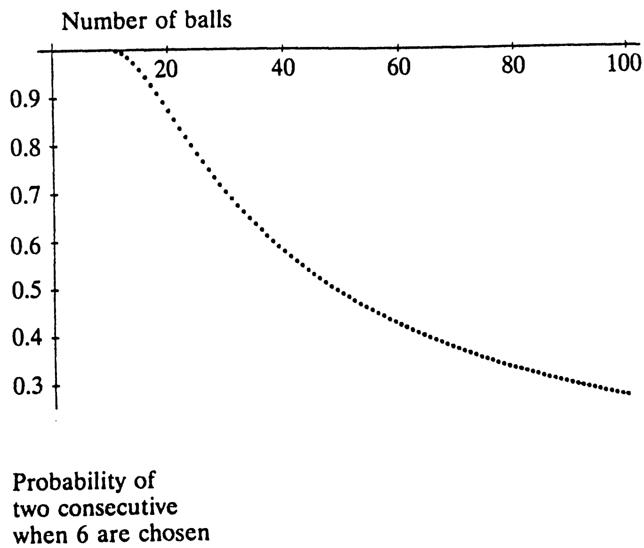

Figure 1

(*Caution*: The device of first putting a ball into each appropriate box works here, where all the balls are identical, but is often misapplied in more general contexts by beginning combinatorics students.)

It is commonly noted that a counting problem may have two solutions, a hard one and an easy one. Often the easy solution is to set up a one-to-one correspondence between what you want to count and something that is easier to count. Can that be done in this case?

Notice that the number of ways to choose k numbers from n with no two consecutive is the same as the number of ways to choose k from $n - k + 1$ without restriction. We can set up a one-to-one correspondence between the two problems as follows:

Suppose the numbers $a_1 < a_2 < \cdots < a_k$ have been chosen from $\{1, 2, \ldots, n-k+1\}$. Then the numbers $a_1, a_2+1, a_3+2, \ldots, a_k+(k-1)$ will be k numbers from $\{1, 2, \ldots, n\}$ with no two consecutive.

Conversely, suppose the numbers $b_1 < b_2 < \ldots < b_k$ have been chosen from $\{1, 2, \ldots, n\}$ with no two consecutive. Then the numbers $b_1, b_2-1, b_3-2, \ldots, b_k-k+1$ will be k distinct numbers chosen from $\{1, 2, \ldots, n-k+1\}$.

This solution is due to Lovász [**2**]; another solution, found in Ryser [**3**], uses a recursive argument.

References

1. B. W. Jackson and Dmitri Thoro, *Applied Combinatorics with Problem Solving*, Addison-Wesley, Reading, MA, 1990.
2. László Lovász, *Combinatorial Problems and Exercises*, North-Holland, Amsterdam, 1979, p. 17.
3. H. J. Ryser, *Combinatorial Mathematics*, Mathematical Association of America, Washington, DC, 1963, pp. 33.
4. Alan Tucker, *Applied Combinatorics*, Wiley, New York, 1980.

Building an Author's Self-Esteem

I think I must apologize to the reader for having published a memoir which is so drearily, I may even say so incomprehensibily written....

I do not want to dispute the mathematical capacity of Mr. Liouville. But, what purpose does such a capacity serve if it is not accompanied by the art of arranging the material and by the art of writing something that can be read, understood and enjoyed. Unfortunately, there are to-day too many young persons, otherwise of a great merit, who consider as an almost immaterial accessory this art which I regard as the essential merit, the merit par excellence for the lack of which all the rest is worthless.

I strongly hope that Mr. Liouville will soon have his revenge for this rather hard reproach, which I have regrettably felt obliged to make to-day, by publishing some memoirs which can be read almost as one reads a novel. However, to be honest I hope this more than I believe it. Long experience has shown me that the illness he suffers from is almost incurable.

Extract from an editorial note by Joseph D. Gergonne, concluding Joseph Liouville's "Mémoire sur la thèorie analytique de la chaleur." From Jesper Lützen, *Joseph Liouville* 1809–1882, Springer-Verlag, New York, 1990, pp. 20–21.