CLASSROOM CAPSULES

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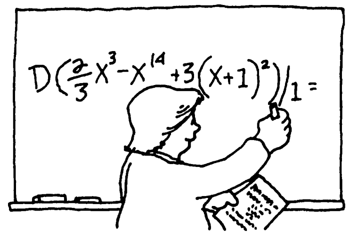

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

The Mathematics of *Cootie*

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Cootie is a children's game in which each player tries to be the first to construct a "cootie bug" from its component parts (Milton Bradley Co., Springfield, MA, 1986). Each part is obtained by tossing a die. The bug's body (which requires tossing a 1) must be gotten first, then the head (a 2 toss). After the player has the body and the head, the remaining parts can be collected in any order: the two eyes (each needs a 3), one nose (a 4), two antennae (each a 5), and the six legs (one 6 apiece). On average, how many tosses of the die does it take to make a cootie?

To find out, we can play the game by using a die and average the number of die tosses, or we can write a computer simulation that will play the game many times and give us the average results. These experimental methods find the expected value to be around 50 tosses. But what about the *theoretical* expected value of the number of tosses it takes to make a cootie? We will explore that question here.

First, suppose that we perform independent trials, each having probability p of success, where $0 < p < 1$, until a total of r successes is accumulated. If we let X be the number of trials required, then the random variable X has a *negative binomial* distribution. The probability that we need n trials to get r successes is given by

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \qquad n = r, r+1, \ldots. \tag{1}$$

Now before we can go on, we must analyze the problem of tossing a 1 to get the body. Letting X be the number of trials required for getting a 1, then the negative binomial formula with $r = 1$ and $p = \frac{1}{6}$ gives

$$P(X = n) = \frac{1}{6}\left(1 - \frac{1}{6}\right)^{n-1}, \qquad n = 1, 2, \ldots. \tag{2}$$

Hence, the expected number of tosses is

$$E(X) = \sum_{n=1}^{\infty} n \frac{1}{6}\left(1 - \frac{1}{6}\right)^{n-1} = \frac{1}{\frac{1}{6}} = 6. \tag{3}$$

Likewise, the expected number of tosses to obtain a 2 for the cootie head is 6. Therefore the expected number of tosses to get the body and then the head is 12.

Having obtained the body and the head, we now want to get two eyes, one nose, two antennae, and six legs—in any order. That is, we need to toss two 3's, one 4, two 5's, and six 6's in any order. Let X be the number of tosses until we have all those outcomes. Then X is a random variable with possible values $11, 12, \ldots$. We want to find the probability distribution of X, $P(X = n)$, for $n = 11, 12, \ldots$. There are four cases, depending on whether the nth toss results in an eye, nose, antenna, or leg to finish the cootie.

Case 1: Gimme an eye. If the outcome of the nth toss is a 3, then the first $n-1$ tosses have exactly one 3, at least one 4, at least two 5's, at least six 6's, and all other tosses are 1 or 2. Let's assume that the number of 5's is i, the number of 4's is j, and the number of 6's is k. The remaining $n-2-i-j-k$ will be 1 or 2. The probability of obtaining these numbers is given by

$$\frac{(n-1)!}{1!\,i!\,j!\,k!(n-2-i-j-k)!}\left(\frac{1}{6}\right)^{2+i+j+k}\left(\frac{2}{6}\right)^{n-2-i-j-k} \tag{4}$$

for all $i \geq 2$, $j \geq 1$, $k \geq 6$, $n \geq 11$, and $(i+j+k) \leq n-2$.

To verify expression (4), note the probability that any sequence of outcomes for the $n-1$ experiments leads to outcome 3 occurring one time, outcome 4 occurring j times, outcome 5 occurring i times, outcome 6 occurring k times, and outcome 1 or 2 occurring $n-2-i-j-k$ times. By the assumed independence of experiments, this probability is $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)^{i}\left(\frac{1}{6}\right)^{j}\left(\frac{1}{6}\right)^{k}\left(\frac{2}{6}\right)^{n-2-i-j-k}$.

How many such outcomes are there? There are $[(n-1)!]/[1!i!j!k!(n-2-i-j-k)!]$ different permutations of $(n-1)$ things where 1 is alike, i are alike, j are alike, k are alike, and $(n-2-i-j-k)$ are alike. Finally, notice that in this case the nth toss should be 3, which occurs with probability $\frac{1}{6}$. Thus (4) is established.

Case 2: Won by a nose. What if the number of the nth toss is 4? By an argument similar to that of case 1, the probability of finishing with a 4 and having i 3's, j 5's, k 6's, and $n-1-i-j-k$ 1's and 2's is

$$\frac{(n-1)!}{i!\,j!\,k!(n-1-i-j-k)!}\left(\frac{1}{6}\right)^{1+i+j+i}\left(\frac{2}{6}\right)^{n-1-i-j-k} \tag{5}$$

for all $i \geq 2$, $j \geq 2$, $k \geq 6$, $n \geq 11$, and $(i+j+k) \leq n-1$.

Case 3: An antenna. If the number of the nth toss is 5, then the analysis is identical to case 1.

Case 4: The last leg. If the number of the nth toss is 6, then by similar reasoning, the probability of each outcome is given by

$$\frac{(n-1)!}{5!\,i!j!\,k!(n-6-i-j-k)!}\left(\frac{1}{6}\right)^{6+i+j+k}\left(\frac{2}{6}\right)^{n-6-i-j-k} \tag{6}$$

for all $i \geq 2$, $j \geq 2$, $k \geq 1$, $n \geq 11$, and $(i+j+k) \leq n-6$.

Combining the four cases, we obtain the probability distribution for this random variable X:

$$P(X=n)=$$

$$\sum_{i=2}^{n-9}\sum_{j=1}^{n-i-8}\sum_{k=6}^{n-2-i-j}\frac{(n-1)!}{1!\,i!\,j!\,k!(n-2-i-j-k)!}\left(\frac{1}{6}\right)^{2+i+j+k}\left(\frac{2}{6}\right)^{n-2-i-j-k}$$

$$+\sum_{i=2}^{n-9}\sum_{j=2}^{n-i-7}\sum_{k=6}^{n-1-i-j}\frac{(n-1)!}{i!\,j!\,k!(n-1-i-j-k)!}\left(\frac{1}{6}\right)^{1+i+j+k}\left(\frac{2}{6}\right)^{n-1-i-j-k}$$

$$+\sum_{i=2}^{n-9}\sum_{j=1}^{n-i-8}\sum_{k=6}^{n-2-i-j}\frac{(n-1)!}{1!\,i!\,j!\,k!(n-2-i-j-k)!}\left(\frac{1}{6}\right)^{2+i+j+k}\left(\frac{2}{6}\right)^{n-2-i-j-k}$$

$$+\sum_{i=2}^{n-9}\sum_{j=2}^{n-i-7}\sum_{k=1}^{n-6-i-j}\frac{(n-1)!}{5!\,i!\,j!\,k!(n-6-i-j-k)!}\left(\frac{1}{6}\right)^{6+i+j+k}\left(\frac{2}{6}\right)^{n-6-i-j-k} \qquad (7)$$

for $n=11,12,\ldots$.

Recall that the expected number of tosses required to earn the body and head is 12. Now we want to add to this the expected value of the probability distribution X represented by (7). Thus, the expected number of tosses to build the entire cootie is

$$12+E(X)=12+\sum_{n=11}^{\infty}nP(X=n).$$

By tailoring a *Mathematica* program, we were able to calculate this value and found it to be 48.953478 (with eight significant digits), consistent with the value obtained in the simulations.

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Minimal Pyramids

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Here is a natural optimization problem that might be given to the better students of a first-year calculus course. Some characteristics of its solution are mildly surprising.

> *Find the dimensions of the pyramid of minimum volume whose base is a regular n-gon and whose base and triangular faces are all tangent to a fixed sphere.*

Figure 1 shows a slice of the pyramid through the apex and perpendicular to an edge of the base polygon. The radius of the inscribed sphere is r, the height of the pyramid is h, and a is the apothem—the distance from the center to the midpoint of a side of the base.

We wish to express the volume of the pyramid in terms of the constant r and the variable h.

Using similar triangles, we have

$$\frac{a}{h}=\frac{r}{\sqrt{h(h-2r)}}$$