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Elementary Linear Algebra and the Division Algorithm

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The set of polynomials is perhaps the first meaningful example of a vector space, appearing in all college textbooks in linear algebra, whose elements are familiar objects to any high school student. However, the old well known properties (of the ring) of polynomials, thoroughly explored in high school, are in general not related, in any convincing way, to the new structure of vector space. This suggests to the student the erroneous idea that no relevant link does exist between those two structures.

The purpose of this note is to exhibit an elementary example of such a link. Namely, we shall describe how to derive, from the simplest properties of the finite dimensional vector spaces, the

Division Algorithm. Let $\mathbb{K}$ be a field and $\mathbb{K}[x]$ be the ring of polynomials, in one indeterminate, with coefficients in $\mathbb{K}$. If $F \in \mathbb{K}[x]$ is not the zero polynomial, then, given $G \in \mathbb{K}[x]$ there exist unique polynomials $Q, R \in \mathbb{K}[x]$ satisfying:

(i) $G = Q \cdot F + R$

(ii) either $R = 0$ or degree $(R) <$ degree (F).

Towards the establishment of this result we shall only make use of the following facts:

(1) In an n-dimensional vector space E, any linearly independent set of n vectors constitutes a basis of E.

(2) The set $\mathcal{P}_n$ of polynomials in $\mathbb{K}[x]$ of degree $d \leq n$ is a vector space of dimension $n+1$ over $\mathbb{K}$.

(3) Any set of nonzero polynomials in $\mathcal{P}_n$ of distinct degrees is linearly independent.

Now, the proof is carried out as follows:

Let $m =$ degree(F) and $n =$ degree(G). If $n < m$ there is nothing to prove since, $G = O \cdot F + G$ is the only possible way to write down G satisfying (i) and (ii) above. Hence, we shall suppose in the sequel that $n \geq m$.

The set $\mathcal{B} = \{1, x, \ldots, x^{m-1}, F, xF, \ldots, x^{n-m}F\}$ of $n+1$ polynomials in $\mathcal{P}_n$ is a basis of $\mathcal{P}_n$. In fact, $\mathcal{B}$ is a linearly independent set because of (3) and consequently a basis in view of (2) and (1). Since $G \in \mathcal{P}_n$ there exist unique scalars $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_{n-m}$ such that

$$G = a_0 \cdot 1 + a_1 x + \cdots + a_{m-1}x^{m-1} + b_0 F + b_1 x F + \cdots + b_{n-m}x^{n-m}F$$
$$= a_0 + a_1 x + \cdots + a_{m-1}x^{m-1} + (b_0 + b_1 x + \cdots + b_{n-m}x^{n-m})F.$$

In other words, $G = QF + R$ as desired. The uniqueness is a direct consequence of the uniqueness of the scalars a_i and b_j ($i = 0, \ldots, m-1$ and $j = 0, \ldots, n-m$).

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Some Calculus-Based Observations Concerning the Solutions to $x'' - q(t)x = 0$

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In this note, using fundamental calculus tools, we will study some of the properties of solutions to the second order linear differential equation

$$x'' - q(t)x = 0 \tag{1}$$

where q is a continuous function for $t \geq 0$ and $q(t) > q_0 > 0$. Specifically, we will show there is a solution that is unbounded and one that is bounded and square-integrable (i.e., $\int_0^\infty x(t)^2\,dt < \infty$). This means that the operator D defined by $Dx = x'' - q(t)x$ is what is known as a limit-point operator. When both solutions are square-integrable, the operator is called limit-circle (see [**1**, Ch. 9] and [**2**] for a further discussion of limit-point and limit-circle operators and their relationship to the spectral theory of differential equations).

Example. Consider the case where $q(t) = a^2$ ($a > 0$). It is easy to see that the equation $x'' - a^2 x = 0$ has the linearly independent solutions $x(t) = \exp(at)$ and $y(t) = \exp(-at)$ where x is unbounded and y is square-integrable.

We begin our discussion by noting that given equation (1) with initial conditions $x(0) = x_0 > 0$ and $x'(0) = \dot{x}_0 > 0$ the solution x and its derivative are both unbounded. In particular, from the initial conditions we must have $x(t) \geq \dot{x}_0 t + x_0$ and $x'(t) \geq q_0 t + \dot{x}_0$ from our assumptions.

Using the unbounded solution x we construct a new solution y by defining (for a derivation see [**3**, pp. 87–91]):

$$y(t) = Ax(t) - x(t)\int_0^t x(s)^{-2}\,ds \qquad \left(A = \int_0^\theta x(t)^{-2}\,dt\right). \tag{2}$$

Observe that A is finite because $\int_0^\infty 1/x(t)^2\,dt \leq \int_0^\infty 1/(x_0 + \dot{x}_0 t)^2\,dt = 1/x_0$ and then it is easy to verify that y is a solution to (1). We claim that y is bounded and square-integrable. First, we see $y(0) = Ax(0) = Ax_0$ and $y(t)$ is always positive by our construction. Furthermore, as $t \to \infty$ we have $y(t) \to 0$. In order to prove this, we rewrite $y(t)$ as

$$y(t) = \left(A - \int_0^t x(s)^{-2}\,ds\right) \Big/ (1/x(t)). \tag{3}$$

Since both numerator and denominator approach 0 as $t \to \infty$, we may apply L'Hôpital's rule to (3) which yields

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} -x(t)^{-2}/(-x'(t)x(t)^{-2}) = \lim_{t\to\infty} 1/x'(t) = 0.$$

Finally, $y'(t)$ must be negative and increasing for all $t = 0$ because, since $y''(t) > 0$, a non-negative value of $y'(t)$ at, say, $t = c$, would imply $y'(t)$ is positive and $y(t)$ is increasing for $t > c$. However, this is impossible since $\lim_{t\to\infty} y(t) = 0$. Thus, y is positive, decreasing, and $0 < y(t) < Ax_0$ for all $t = 0$.

To show y is square-integrable, start with the identity $0 = y'' - q(t)y$, multiply by y, and integrate from 0 to t using integration by parts on the first term. So

$$0 = \int_0^t yy''\,ds - \int_0^t qy^2\,ds = y(t)y'(t) - y(0)y'(0) - \int_0^t y'^2\,ds - \int_0^t qy^2\,ds$$

or

$$\int_0^t y'(s)^2\,ds + \int_0^t q(s)y(s)^2\,ds = y(t)y'(t) - y(0)y'(0) \tag{4}$$

and this equation must hold for all $t = 0$. Since $\lim_{t\to\infty} y(t)y'(t) = 0$ and the terms on the left side of (4) are both positive, we must have

$$\int_0^\infty \left|y'(t)\right|^2\,dt + \int_0^\infty q(t)\,|y(t)|^2\,dt = -y(0)y'(0) \tag{5}$$

and both y and y' are square-integrable.

Remark 1. By construction, $y(0) = Ax(0)$ and $y'(0) = Ax'(0) - 1/x(0)$, where the latter value must be negative. Also, note that $q(t)$ need not be differentiable on [0, 8). Lastly, relationship (4) is valid as long as $q(t) > 0$. However, the bounded solution need not be square-integrable . For example, when $q(t) = 1/(4(t+1)2)$, then any non-trivial bounded solution is of the form $y(t) = b(t+1)^{(1/2)(1-\sqrt{2})}$ where b is any non-zero real constant. This function is obviously not square-integrable. However, its derivative is square-integrable from (5).

Remark 2. We have also shown that any bounded solution must approach 0 as $t \to \infty$ because the solution space consists of linear combinations of one bounded and one unbounded solution. Therefore, all bounded solutions must be of the form $cy(t)$ where c is any real number.

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