3. Graph the function $(1/25)P$, where P is the price function from part 1. Since the car gets 25 miles per gallon, $(1/25)P(x)$ is your cost per mile during the first x miles if you fill the tank x miles from city A. Find a geometric interpretation of the total cost of gasoline as the sum of areas of rectangles overlaid on this graph. Repeat for the price function of part 2.
4. Now repeat parts 1 and 2 allowing two stops between A and B. Is this cheaper? What is the least possible cost if there is no limit on the number of stops? Show the geometric interpretation as areas for the case of unlimited stops.

After the problem was assigned, we spent a few minutes of the following class meetings discussing the students' progress and giving hints. It was surprising to see how many students had trouble writing down the linear expression for the price function and finding the values of p and k without help. Although only about half of the students got the point in part 4 about unlimited stops and integration, many of those who did confessed to being fascinated by it. Some also appreciated using *Maple* in the context of a larger "real" problem rather than just another drill. In the end-of-semester critiques, many students mentioned that they like doing "real-world" problems such as the rental car problem. One even claimed to have used these ideas to save money on the trip home during a break.

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Complex Eigenvalues and Rotations: Are Your Students Going in Circles?

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In an elementary linear algebra class, when you encounter a real matrix with complex eigenvalues, what do you say? Do you comment that it represents essentially a rotation in an unusual coordinate system? This approach is well explained in D. Lay's *Linear Algebra and Its Applications* (Addison Wesley, Reading, MA, 1994), where one finds this theorem.

Theorem. *Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and associated eigenvector $\mathbf{w}$ in $\mathbb{C}^2$. Then $A = PCP^{-1}$, where P is the 2×2 real matrix $[\mathrm{Re}(\mathbf{w})\ \ \mathrm{Im}(\mathbf{w})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.*

If $|\lambda| = 1$, then $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the standard matrix of a rotation by the angle θ; otherwise C represents the composition of this rotation with a scaling by the factor $|\lambda|$. Since the matrix P is the standard matrix of the linear transformation τ that sends the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to the basis $\{\mathbf{u}, \mathbf{v}\} = \{\mathrm{Re}(\mathbf{w}), \mathrm{Im}(\mathbf{w})\}$, it follows that A is the standard matrix of the composition of τ^{-1}, then the rotation through the angle θ, then the scaling by the factor $|\lambda|$, and finally τ. Without further information about the basis $\{\mathbf{u}, \mathbf{v}\}$, however, it is hard to picture the geometric effect of τ and consequently of A.

A 3-D perspective. Perhaps we can do better by considering the 3×3 matrix $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ and the associated transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$. In fact, we will show that

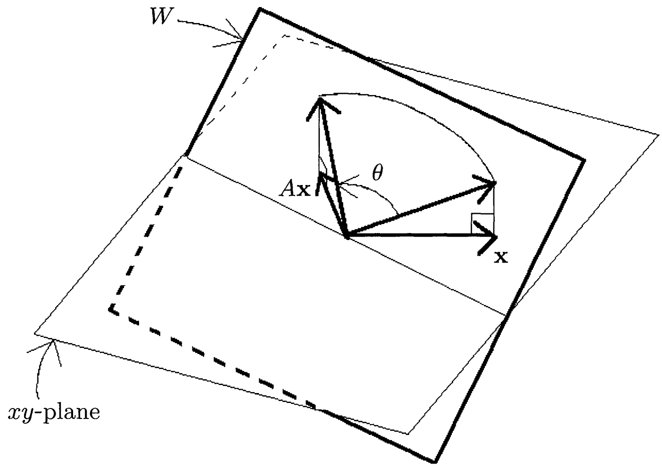

Figure 1

there is a plane W through the origin in $\mathbb{R}^3$ such that the geometric effect of α on a vector $\mathbf{x}$ in the xy-plane is the composition of a vertical lifting of $\mathbf{x}$ to the plane W, rotation in W through the angle θ, scaling by $|\lambda|$ in W, and finally projection back to the xy-plane. See Figure 1, where the case $|\lambda| = 1$ is diagrammed.

In order to construct the rotation plane W, we will use the following lemma.

Lemma. *Let A be a real matrix with nonreal eigenvalue λ. There is a corresponding complex eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are orthogonal real vectors with $|\mathbf{u}| = 1$ and $|\mathbf{v}| \leq 1$.*

Proof. Let $\mathbf{w}_0$ be any complex eigenvector corresponding to λ. Write $\mathbf{w}_0 = \mathbf{u}_0 + i\mathbf{v}_0$ where $\mathbf{u}_0$ and $\mathbf{v}_0$ are real. Since every nonzero complex scalar multiple of $\mathbf{w}_0$ is also an eigenvector corresponding to λ, we may assume that $|\mathbf{u}_0| \geq |\mathbf{v}_0|$; for if this were not the case we could just replace $\mathbf{w}_0$ by the eigenvector $i\mathbf{w}_0 = -\mathbf{v}_0 + i\mathbf{u}_0$. Now if $(\mathbf{u}_0, \mathbf{v}_0) = 0$, then $\mathbf{w} = \mathbf{w}_0/|\mathbf{u}_0|$, $\mathbf{u} = \mathbf{u}_0/|\mathbf{u}_0|$, and $\mathbf{v} = \mathbf{v}_0/|\mathbf{u}_0|$ satisfy the required conditions.

Otherwise, consider

$$\begin{aligned}\mathbf{w}_1 &= (\cos\varphi + i\,\sin\varphi)\mathbf{w}_0 \\ &= (\cos\varphi\mathbf{u}_0 - \sin\varphi\mathbf{v}_0) + i(\sin\varphi\mathbf{u}_0 + \cos\varphi\mathbf{v}_0) = \mathbf{u}_1 + i\mathbf{v}_1.\end{aligned}$$

The inner product of the real and imaginary parts of the eigenvector $\mathbf{w}_1$ is

$$\begin{aligned}(\mathbf{u}_1, \mathbf{v}_1) &= (\cos\varphi\mathbf{u}_0 - \sin\varphi\mathbf{v}_0, \sin\varphi\mathbf{u}_0 + \cos\varphi\mathbf{v}_0) \\ &= \sin\varphi\cos\varphi\,[(\mathbf{u}_0, \mathbf{u}_0) - (\mathbf{v}_0, \mathbf{v}_0)] + \left(\cos^2\varphi - \sin^2\varphi\right)(\mathbf{u}_0, \mathbf{v}_0) \\ &= \frac{1}{2}\sin 2\varphi\left(|\mathbf{u}_0|^2 - |\mathbf{v}_0|^2\right) + \cos 2\varphi\,(\mathbf{u}_0, \mathbf{v}_0).\end{aligned}$$

Since $(\mathbf{u}_0, \mathbf{v}_0) \neq 0$, we may choose $\varphi = \frac{1}{2}\operatorname{arccot}\left(\frac{|\mathbf{v}_0|^2 - |\mathbf{u}_0|^2}{2(\mathbf{u}_0, \mathbf{v}_0)}\right)$. Then $(\mathbf{u}_1, \mathbf{v}_1) = 0$, and (replacing $\mathbf{w}_1$ by $i\mathbf{w}_1$ if necessary) we may assume that $|\mathbf{u}_1| \geq |\mathbf{v}_1|$. Thus $\mathbf{w} = \mathbf{w}_1/|\mathbf{u}_1|$, $\mathbf{u} = \mathbf{u}_1/|\mathbf{u}_1|$, and $\mathbf{v} = \mathbf{v}_1/|\mathbf{u}_1|$ satisfy the conditions of the lemma.

Let A be a 2×2 real matrix with a nonreal eigenvalue $\lambda = \cos\theta - i\sin\theta$. Let $\mathbf{u} + i\mathbf{v}$ be a corresponding eigenvector with $\mathbf{u}$ and $\mathbf{v}$ real vectors, as in the lemma, with $|\mathbf{u}| = 1$ and $|\mathbf{v}| \leq 1$. Then $A(\mathbf{u} + i\mathbf{v}) = (\cos\theta - i\sin\theta)(\mathbf{u} + i\mathbf{v})$. If we look at the real and imaginary parts in this equation we obtain

$$\begin{aligned} A\mathbf{u} &= \cos\theta\mathbf{u} + \sin\theta\mathbf{v} \\ A\mathbf{v} &= -\sin\theta\mathbf{u} + \cos\theta\mathbf{v}. \end{aligned} \tag{1}$$

Note that from the second of these equations it follows that $\mathbf{v}$ cannot be zero, for $-\sin\theta$, the imaginary part of λ, is nonzero by assumption. We may assume that in $\mathbb{R}^3$ the orthogonal vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{e}_3$ form a right-handed system. Otherwise we could replace λ by its conjugate, which is also an eigenvalue; in effect this would replace $\mathbf{v}$ by $-\mathbf{v}$ and θ by $-\theta$. Choose a first quadrant angle β such that $\cos\beta = |\mathbf{v}|$ and, in $\mathbb{R}^3$, let $\mathbf{v}_1 = \mathbf{v} + \sin\beta\mathbf{e}_3$. Then $|\mathbf{v}_1| = 1$ and $\mathbf{v}_1$ is orthogonal to $\mathbf{u}$. Let W be the plane through the origin spanned by $\mathbf{u}$ and $\mathbf{v}_1$. This plane intersects the xy-plane in a line containing $\mathbf{u}$ and makes the angle β with the xy-plane (see Figure 2).

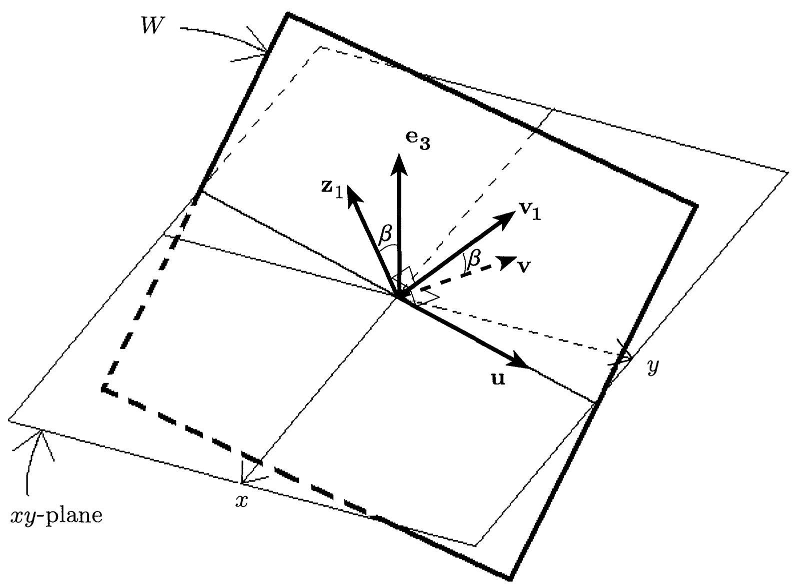

Figure 2

Define $\mathbf{z}_1 = \mathbf{u} \times \mathbf{v}_1$. Then $[\mathbf{u},\ \mathbf{v},\ \mathbf{e}_3]$ is an orthogonal basis for $\mathbb{R}^3$ and $[\mathbf{u},\ \mathbf{v}_1,\ \mathbf{z}_1]$ is an orthonormal basis for $\mathbb{R}^3$. We can define a linear transformation on $\mathbb{R}^3$ by specifying the action of the transformation on these bases. Let π be the linear transformation such that

$$\pi(\mathbf{u}) = \mathbf{u} \qquad \pi(\mathbf{v}) = \mathbf{v}_1 \qquad \pi(\mathbf{e}_3) = \mathbf{z}_1.$$

Note that a point in the xy-plane of $\mathbb{R}^3$ has the form $a\mathbf{u} + b\mathbf{v} + 0\mathbf{e}_3$ and is mapped by π to $a\mathbf{u} + b\mathbf{v}_1 + 0\mathbf{z}_1 = a\mathbf{u} + b\mathbf{v} + b\sin\beta\mathbf{e}_3$, so π is a vertical lifting of points in the xy-plane to points in W. (Its restriction to the subspace orthogonal to $\mathbf{u}$ is a rotation through the angle β followed by stretching along the line of $\mathbf{v}_1$ by the factor $\sec\beta$; however, we are interested only in the effect of π on the xy-plane.)

Let ρ denote rotation in $\mathbb{R}^3$ through the angle θ about $\mathbf{z}_1$ as an axis. (The rotation should be counterclockwise as seen from the tip of $\mathbf{z}_1$ looking back toward W.) Then

$$\rho(\mathbf{u}) = \cos\,\theta\mathbf{u} + \sin\,\theta\mathbf{v}_1$$
$$\rho(\mathbf{v}_1) = -\sin\,\theta\mathbf{u} + \cos\,\theta\mathbf{v}_1$$
$$\rho(\mathbf{z}_1) = \mathbf{z}_1.$$

The transformation π has an inverse since it carries a basis to a basis. Observe that the transformation $\alpha = \pi^{-1}\rho\pi$ satisfies

$$\alpha(\mathbf{u}) = \pi^{-1}\rho(\mathbf{u}) = \pi^{-1}(\cos\,\theta\mathbf{u} + \sin\,\theta\mathbf{v}_1) = \cos\,\theta\mathbf{u} + \sin\,\theta\mathbf{v}$$
$$\alpha(\mathbf{v}) = \pi^{-1}\rho(\mathbf{v}_1) = \pi^{-1}(-\sin\,\theta\mathbf{u} + \cos\,\theta\mathbf{v}_1) = -\sin\,\theta\mathbf{u} + \cos\,\theta\mathbf{v}$$
$$\alpha(\mathbf{e}_3) = \pi^{-1}\rho(\mathbf{z}_1) = \pi^{-1}(\mathbf{z}_1) = \mathbf{e}_3.$$

These equations with (1) demonstrate that the matrix representing the transformation $\alpha = \pi^{-1}\rho\pi$ with respect to the basis $[\mathbf{u},\ \mathbf{v},\ \mathbf{e}_3]$ is $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. This verifies that the action of A can be described geometrically as lifting vectors vertically from the xy-plane to the plane W, rotating in W, and returning vertically to the xy-plane.

As an example [see Lay, p. 308] let's consider the matrix $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$, which has eigenvalue $\lambda = 0.8 - 0.6i$, with an associated eigenvector $\begin{bmatrix} -2 \\ 5 \end{bmatrix} + i \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. Let $C = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$. Then C is a rotation matrix for the angle $\theta = \arctan(0.6/0.8)$ and $A = PCP^{-1}$, as asserted by the theorem cited earlier.

The calculations suggested by the lemma yield $\varphi = \frac{1}{2}\text{arccot}\left(\frac{16-29}{16}\right) \approx 1.1266$. So the new eigenvector with orthogonal real and imaginary parts is

$$(\cos\,\varphi + i\,\sin\,\varphi)\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix} + i\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2.7522 \\ 2.1489 \end{bmatrix} + i\begin{bmatrix} -3.5250 \\ 4.5147 \end{bmatrix}.$$

Since its imaginary part has greater magnitude than its real part, this vector must be multiplied by i; scaling then yields the eigenvector $\mathbf{u}+i\mathbf{v} = \begin{bmatrix} 0.6154 \\ -0.7882 \end{bmatrix} + i\begin{bmatrix} 0.4805 \\ 0.3752 \end{bmatrix}$ with $|\mathbf{u}| = 1$ and $|\mathbf{v}| = \cos\,\beta \approx 0.6096 \leq 1$. So the rotation plane W is spanned by the vectors $\mathbf{u} = [0.61540\ \ -0.7882\ \ 0]^T$ and $\mathbf{v}_1 = \mathbf{v} + \sin\beta\mathbf{e}_3 = [0.4805\ \ 0.3752\ \ 0.7927]^T$, and it makes an angle of $\beta \approx 0.9152$, or about 52°, with the xy-plane.

These results easily generalize to the case where A is a 3×3 matrix, with suitable adjustment for the third (real) eigenvalue. Several related questions may interest students. What is a simple test to detect matrices that represent rotations? This is elementary for 2×2 matrices but is more interesting for 3×3 matrices. How do you identify the axis of rotation for a 3×3 rotation matrix? If you randomly generate a 2×2 real matrix, what is the probability that its eigenvalues will be nonreal?

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