A Direct Proof of the Integral Formula for Arctangent

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In this capsule, we give a direct proof that the Arctangent is an integral of $1/(1+x^2)$. It then becomes possible to use the Arctangent to determine the tangent and the other trigonometric functions. Here (Figure 1) for any real number a, we define Arctan a as the angle θ (in radians) determined by angle OPR, where θ is taken as negative if $a<0$.

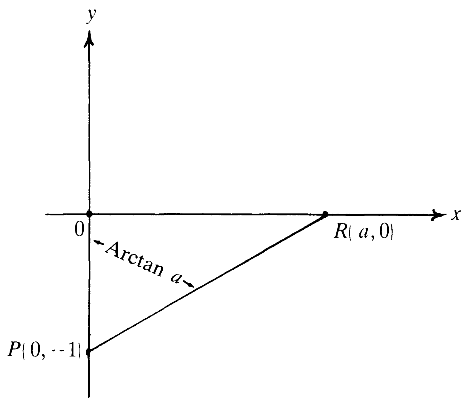

Figure 1.

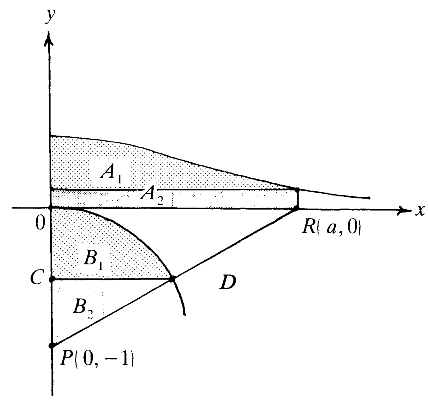

Figure 2.

In what follows, we fix a number $a>0$. This will determine two regions, as shown in Figure 2. The region above the x-axis is bounded by the graph of $y = \dfrac{1}{2(1+x^2)}$ and the x-axis, where $0 \le x \le a$. Therefore, the total area of this region is

$$\int_0^a \frac{dx}{2(1+x^2)}.$$

The region below the x-axis is a sector of a circle having center $(0,-1)$ and radius 1. The sides of the sector are determined by the y-axis and the line connecting $(0,-1)$ to $(a,0)$. Thus, these sides determine the angle with value Arctan a. Since the area of a sector of a circle of radius r and angle θ (in radians) is $r^2\theta/2$, the total area of this shaded region is $\frac{1}{2}$ Arctan a.

We shall show that these two shaded regions have equal areas. From this, it follows that the Arctangent can be represented as an integral of the function $y=1/(1+x^2)$.

First, consider the region above the x-axis (Figure 2). This region is divided into two subregions, A_1 and A_2. The rectangle A_2 has area $\mathscr{A}(A_2) = \dfrac{a}{2(1+a^2)}$.

The shaded sector below the x-axis is also divided into two subregions, B_1 and B_2. Since triangle CPD is similar to triangle OPR, the legs PC and CD of triangle CPD have lengths $1/\sqrt{1+a^2}$ and $a/\sqrt{1+a^2}$, respectively. Thus, B_2 has area $\mathscr{A}(B_2) = \dfrac{a}{2(1+a^2)}$. In particular, $\mathscr{A}(A_2) = \mathscr{A}(B_2)$.

It remains to be shown that $\mathscr{A}(A_1)=\mathscr{A}(B_1)$. First, solve the equation $y = 1/2(1+x^2)$ for x to obtain $x=\sqrt{1-2y}/\sqrt{2y}$. Then integrate this along the y-axis to obtain

$$\mathscr{A}(A_1)=\int_{1/2(1+a^2)}^{1/2}\frac{\sqrt{1-2y}}{\sqrt{2y}}\,dy.$$

Likewise, the circular boundary of B_1 can be represented as the graph of $x = \sqrt{1-(y+1)^2}$, where $\frac{1}{\sqrt{1+a^2}}-1\le y\le 0$. Therefore,

$$\mathscr{A}(B_1)=\int_{\frac{1}{\sqrt{1+a^2}}-1}^{0}\sqrt{1-(y+1)^2}\,dy.$$

Finally, we show that the integral for $\mathscr{A}(A_1)$ can be transformed into the integral for $\mathscr{A}(B_1)$ by means of the substitution $t=\sqrt{2y}-1$. Indeed, $dt=\frac{dy}{\sqrt{2y}}$ and $(t+1)^2=2y$. Therefore,

$$\begin{aligned}\mathscr{A}(A_1)&=\int_{1/2(1+a^2)}^{1/2}\frac{\sqrt{1-2y}}{\sqrt{2y}}\,dy\\&=\int_{(1/\sqrt{1+a^2})-1}^{0}\sqrt{1-(t+1)^2}\,dt\\&=\mathscr{A}(B_1).\end{aligned}$$

We have therefore shown that $\mathscr{A}(A_1)+\mathscr{A}(A_2)=\mathscr{A}(B_1)+\mathscr{A}(B_2)$. Thus,

$$\int_0^a\frac{dx}{2(1+x^2)}=\frac{1}{2}\text{Arctan } a$$

or

$$\int_0^a\frac{dx}{(1+x^2)}=\text{Arctan } a \qquad (*)$$

for $a>0$.

A simple symmetry argument establishes the validity of $(*)$ for $a<0$. Equation $(*)$ is clearly valid for $a=0$. Thus, $(*)$ is valid for all real values a.

We outline a method for obtaining the derivatives of the trigonometric functions from $(*)$. First, apply the fundamental theorem of calculus to obtain the derivative of the Arctangent. The function $f(x)=\tan x$ $(-\pi/2<x<\pi/2)$ is the inverse of the Arctangent, and its derivative $f'(x)=\sec^2 x$ can be obtained from the inverse function theorem. Since the tangent function is a repetition of f on all intervals of the form $((n-\frac{1}{2})\pi,(n+\frac{1}{2})\pi)$, we have

$$\frac{d}{dx}\tan x=\sec^2 x.$$

Next, use the tangent function to represent the secant, and differentiate to obtain the usual formula for the derivative of the secant. For the derivatives of the sine and cosine, observe that $\cos x = 1/(\sec x)$ and $\sin x = \tan x \cos x$ for $x \neq (n + \frac{1}{2})\pi$. Differentiate to obtain the usual formulas with this restriction which can be removed by use of the identities

$$\sin x = \cos\left(x - \frac{\pi}{2}\right) \text{ and } \cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Finally, the derivatives of the cotangent and cosecant can be obtained from the derivatives of the sine and cosine in the usual way.

Riemann Integral of cos x

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Lagrange's identity,

$$\sum_{k=0}^{n-1} \cos k\theta = \frac{1}{2} + \frac{\sin\left(n - \frac{1}{2}\right)\theta}{2 \sin \frac{1}{2}\theta},$$

can be verified using mathematical induction and the trigonometric identity,

$$\sin(u + v) - \sin(u - v) = 2 \cos u \sin v.$$

We can use Lagrange's identity to obtain a basic calculus formula. Since

$$\sum_{k=0}^{n-1} \frac{x}{n} \cos k\frac{x}{n} = \frac{x}{n}\left[\frac{1}{2} + \frac{\sin\left(n - \frac{1}{2}\right)\frac{x}{n}}{2 \sin \frac{x}{2n}}\right]$$

is a Riemann sum for the function $f(t) = \cos t$ on the interval $[0, x]$, we have

$$\int_0^x \cos t\, dt = \lim_{n \to \infty}\left[\frac{x}{2n} + \frac{\frac{x}{2n}}{\sin \frac{x}{2n}} \sin\left(x - \frac{x}{2n}\right)\right] = 0 + (1)(\sin x).$$

That is, we have shown that

$$\int_0^x \cos t\, dt = \sin x$$

without using the fundamental theorem of calculus. Compare [James Stewart, *Calculus*, Brooks/Cole, Monterey, CA, pp. 266–267.]