CLASSROOM CAPSULES

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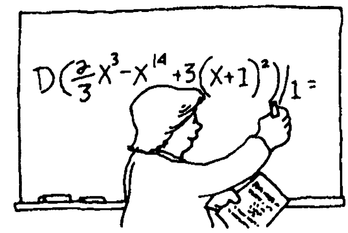

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Cable-laying and Intuition

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In almost every calculus book there appears a problem isomorphic to the following:

> Points A and B are opposite each other on the shore of a straight river that is w feet wide. Point C is on the same side of the river as B, l feet down the river. A telephone company wishes to lay a cable from A to C. It costs \$$a$ per foot to run the cable underwater and \$$b$ per foot to run the cable on land. Which path would be least expensive for the company?

Other versions of the problem have a person walking along the shore and swimming, or rowing a boat, across the river and wanting to minimize the time to get from A to C. All versions lead to the picture in Figure 1.

If $a \leq b$, it is intuitively clear that the least expensive path is the straight line joining A to C. If a is much greater than b, it is plausible to guess that the least expensive path would be to go from A to B and then from B to C; that is, one would think that

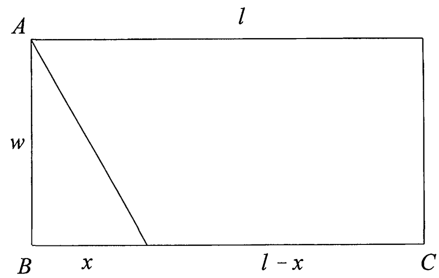

Figure 1.

minimizing the amount of cable to be laid underwater would be best. Surprisingly, this intuition is unreliable: the piecewise linear path joining A to B and B to C is *never* the least expensive path, at least if $b > 0$.

To see this, the problem is to minimize

$$f(x) = a\sqrt{w^2 + x^2} + b(l - x), \quad 0 \le x \le l, \quad a, b, w, l > 0.$$

Computing $f'(x)$ shows that there is exactly one critical value,

$$x_0 = \frac{bw}{\sqrt{a^2 - b^2}},$$

which can occur only when $a > b$. Substituting,

$$f(0) = aw + bl, \quad f(l) = a\sqrt{w^2 + l^2}, \quad f(x_0) = w\sqrt{a^2 - b^2} + bl.$$

If $a \le b$ then there is no critical value and we need only compare $f(0)$ with $f(l)$. Since

$$a^2(w^2 + l^2) < a^2w^2 + 2abwl + a^2l^2 \le (aw + bl)^2,$$

$f(l) < f(0)$, so f is minimized at l. The least expensive path in this case is the linear path joining A and C.

If $a > b$, we must compare $f(0),\ f(x_0),\ \text{and} f(l)$. It is easy to see that

$$f(x_0) = w\sqrt{a^2 - b^2} + bl < w\sqrt{a^2} + bl = f(0)$$

which seems to justify the assertion that f cannot be minimized at 0.

But are we really finished? It is easy to overlook the possibility (and we are willing to bet that many calculus students would be guilty of the oversight) that x_0 might not lie in the domain $0 \le x \le l$ of f. Thus, for f to be minimized at 0, the two conditions $x_0 > l$ and $f(0) \le f(l)$ must be met. The first is

$$\frac{bw}{\sqrt{a^2 - b^2}} > l, \text{ so } w > \frac{l\sqrt{a^2 - b^2}}{b}.$$

The second is

$$aw + bl \le a\sqrt{w^2 + l^2} \text{ or } a^2w^2 + 2abwl + b^2l^2 \le a^2w^2 + a^2l^2, \text{ so } w \le \frac{l(a^2 - b^2)}{2ab.}.$$

These imply

$$\sqrt{a^2 - b^2} < \frac{a^2 - b^2}{2a}.$$

But this says

$$2a\sqrt{a^2 - b^2} < a^2 - b^2, \text{ or } 2a < \sqrt{a^2 - b^2} < \sqrt{a^2} = a,$$

which is impossible.

To summarize, if $0 \le x_0 \le l$, then $f(x_0) < f(0)$; if $x_0 > l$, then $f(l) < f(0)$. Hence, the piecewise linear path joining A to B and B to C is never the least expensive path.

If a/b is very large, then $x_0 = w/\sqrt{(a/b)^2 - 1}$ is close to zero, so the incorrect intuition that f is minimized at 0 becomes correct "in the limit."

It is interesting to note that when $a > b$ the *maximum* of f occurs either at 0 or l: at 0 if $w \geq ((a^2 - b^2)/2ab)l$ and at l if $w \leq ((a^2 - b^2)/2ab)l$.

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Taylor's Formula via Determinants

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For calculus students who know determinants one can, after doing Rolle's theorem, proceed to the following

Theorem. Let $f(x), f_1(x), \ldots, f_{n+2}(x)$ be $n+1$ times continuously differentiable functions. Then

$$\begin{vmatrix} f(x) & f_1(x) & \ldots & f_{n+2}(x) \\ f(0) & f_1(0) & \ldots & f_{n+2}(0) \\ f'(0) & f_1'(0) & & f_{n+2}'(0) \\ & \ldots & & \\ f^{(n)}(0) & f_1^{(n)}(0) & \ldots & f_{n+2}^{(n)}(0) \\ f^{(n+1)}(h) & f_1^{(n+1)}(h) & \ldots & f_{n+2}^{(n+1)}(h) \end{vmatrix} = 0 \qquad (1)$$

for some h between 0 and x.

Proof. Consider x as constant and let $D^{(i)}(h)$ denote the function of h obtained by replacing the last row of the determinant with $f^{(i)}(h)\, f_1^{(i)}(h) \ldots f_{n+2}^{(i)}(h)$. Observe that for $i = 0, 1, \ldots, n$ the derivative of $D^{(i)}(h)$ with respect to h is $D^{(i+1)}(h)$ and the determinant in (1) is $D^{(n+1)}(h)$. Now $D^{(0)}(0) = 0$ because the second and the last rows are the same; likewise, $D^{(0)}(x) = 0$ because its first and last rows are the same. So, by Rolle's theorem, $D^{(1)}(h) = 0$ for some h between 0 and x. Also, the last row of $D^{(1)}(0)$ is the same as its third. So, using Rolle's theorem again, $D^{(2)}(h) = 0$ for some h between 0 and x. Continuing, we see that $D^{(n+1)}(h) = 0$ for a suitable h between 0 and x. *q.e.d.*

For example, (1) shows that for some h between 0 and x, we have

$$\begin{vmatrix} f(x) & 1 & \frac{x}{1!} & \frac{x^2}{2!} & \ldots & \frac{x^n}{n!} & \frac{x^{n+1}}{(n+1)!} \\ f(0) & 1 & 0 & 0 & \ldots & 0 & 0 \\ f'(0) & 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & & \\ f^{(n)}(0) & 0 & 0 & 0 & \ldots & 1 & 0 \\ f^{(n+1)}(h) & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix} = 0$$

which is Taylor's formula because the determinant is

$$f(x) - f(0) - \frac{x}{1!}f'(0) - \frac{x^2}{2!}f''(0) - \cdots - \frac{x^n}{n!}f^{(n)}(0) - \frac{x^{n+1}}{(n+1)!}f^{n+1}(h).$$

