CLASSROOM CAPSULES

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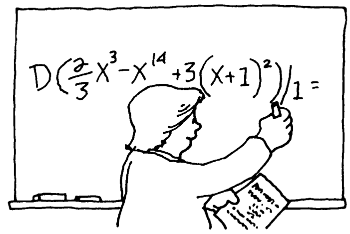

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

Polar Summation

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It is well known that parallel techniques exist between computing double sums and computing double integrals. A familiar example is reversing the order of integration which will sometimes make what appears to be an impossible evaluation into a routine one. In this note we will see that another useful manipulation for computing double integrals has an interesting analogue in double sums. Consider the classic integral $I=\int_{-\infty}^{\infty}e^{-x^2}\,dx$ which at first glance may seem hopeless since e^{-x^2} has no elementary antiderivative. However, upon squaring and transforming the integral from rectangular to polar coordinates we get

$$I^2=\int_{-\infty}^{\infty}e^{-x^2}\,dx\int_{-\infty}^{\infty}e^{-y^2}\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy=\int_{0}^{2\pi}\int_{0}^{\infty}e^{-r^2}r\,dr\,d\theta=\pi.$$

We will use a similar technique to evaluate a sum that was suggested as a problem during a discussion in a number theory course. Evaluate

$$\sum_{a,b\in\mathbb{Z}^+}\frac{1}{[a,b]^2} \tag{1}$$

where $[a,b]$ represents the least common multiple of a and b and $\mathbb{Z}^+$ stands for the positive integers.

To evaluate (1) recall that $[a,b](a,b)=ab$, with (a,b) denoting the greatest common divisor of a and b, and let $r=(a,b)$. If we write $a=\alpha r$ and $b=\beta r$

where $(\alpha, \beta) = 1$, then

$$\sum_{a, b \in \mathbb{Z}^+} \frac{1}{[a, b]^2} = \sum_{a, b \in \mathbb{Z}^+} \frac{(a, b)^2}{(ab)^2} = \sum_{\substack{(\alpha, \beta) = 1 \\ r \in \mathbb{Z}^+}} \frac{r^2}{(r\alpha r\beta)^2} = \sum_{r=1}^{\infty} \frac{1}{r^2} \sum_{(\alpha, \beta) = 1} \frac{1}{\alpha^2 \beta^2}. \tag{2}$$

Since the sum over r is readily recognized as the Riemann zeta function $\zeta(2) = \pi^2/6$ [G. H. Hardy and E. M. Wright, *The Theory of Numbers*, 2nd ed., Oxford University Press, London, 1945], the evaluation of (2) reduces to finding the value of the summation taken over $(\alpha, \beta) = 1$. If we restrict our attention to lattice points in Q_1, the first quadrant, then the points with relatively prime coordinates are geometrically those that are "visible" from the origin. A ray from the origin through a "visible" point (α, β) will pass through all of the "invisible" multiples of it: $(2\alpha, 2\beta), (3\alpha, 3\beta), \ldots$. Now, we can compute $S = \sum_{(a, b) \in Q_1} 1/a^2 b^2$ by summing along each ray (i.e., polar coordinates) to obtain

$$S = \sum_{(\alpha, \beta) = 1} \sum_{r=1}^{\infty} \frac{1}{(r\alpha r\beta)^2} = \sum_{r=1}^{\infty} \frac{1}{r^4} \sum_{(\alpha, \beta) = 1} \frac{1}{\alpha^2 \beta^2} = \zeta(4) \sum_{(\alpha, \beta) = 1} \frac{1}{\alpha^2 \beta^2}. \tag{3}$$

However, by summing over a and b individually (i.e., rectangular coordinates) we see that

$$S = \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left(\sum_{b=1}^{\infty} \frac{1}{b^2} \right) = \zeta^2(2). \tag{4}$$

Using $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$ we find on equating (3) and (4) that

$$\sum_{(\alpha, \beta) = 1} \frac{1}{\alpha^2 \beta^2} = \frac{\zeta^2(2)}{\zeta(4)} = \frac{5}{2}.$$

Thus from (1) and (2) we conclude that

$$\sum_{a, b \in \mathbb{Z}^+} \frac{1}{[a, b]^2} = \frac{5\pi^2}{12}.$$

Finally, as an exercise, use an argument similar to the one above to show that for any positive integer n we have the generalization

$$\sum_{a, b \in \mathbb{Z}^+} \frac{1}{[a, b]^{2n}} = \frac{\zeta^3(2n)}{\zeta(4n)}.$$

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