After utilizing a standard trigonometric substitution, we see that, for $0 \leq \theta < \pi/2$,

$$L(\theta) = \frac{1}{g}\left(v^2 \sin\theta + v^2 \frac{\cos^2\theta}{2} \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right),$$

while the case $\theta = \pi/2$, corresponding to shooting the cannonball straight up, leads to $L(\pi/2) = v^2/g$.

To compute the critical values, we take the derivative of $L(\theta)$ and find

$$L'(\theta) = \frac{1}{g}\left(2v^2\cos\theta - v^2\cos\theta\sin\theta\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right).$$

The critical value for θ in $(0, \pi/2)$ is the angle θ_0 which satisfies

$$\sin\theta_0 \ln\left|\frac{1+\sin\theta_0}{1-\sin\theta_0}\right| = 2.$$

At the endpoints of the interval $[0, \pi/2]$, we see that $L(0) = 0$ and that $L(\pi/2)$ gives a local minimum. Using a graphing calculator, or by Newton's method, one easily obtains $\theta_0 = .985514738\ldots$ radians or about 56 degrees. It is interesting to notice that this θ_0, which maximizes the arclength, lies strictly between the range-maximizing angle $\pi/4$ and the height-maximizing angle $\pi/2$.

Pictures Suggest How to Improve Elementary Numerical Integration

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In a recent introductory numerical methods course, Maple helped students discover some substantial improvements to the trapezoidal and Simpson's methods. Students had learned to do numerical integration using rectangles and trapezoids, and were just starting to use Simpson's formula.

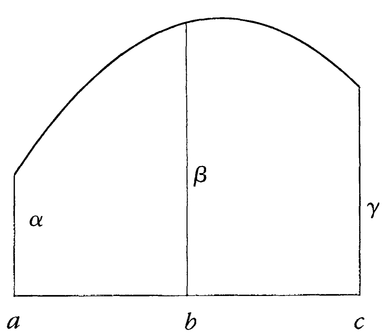

Figure 1

In Figure 1, the base is $(c-a)$, and $\frac{1}{6}(\alpha + 4\beta + \gamma)$ is an average altitude. The formula comes from passing a parabolic arc passing through three points; this arc follows a function's graph more closely than the line segment used in the rectangular or trapezoidal methods. After the class wrote a short computer program using Simpson's formula for approximating integrals, we tested it on $\int_0^2 x^3\,dx$. Students found that the program gave the exact answer of 4, regardless of the number of subdivisions. This struck a number of them as remarkable since Simpson's formula had been developed to be exact only for quadratics, not cubics.

Why does this happen? The graph of $y = x^3$ passes through $(0,0)$, $(1,1)$ and $(2,8)$, and the approximating parabola through these points is $y = 3x^2 - 2x$. As can be seen in Figure 2, the parabola dips below the cubic from $x = 0$ to 1, and rises above it from $x = 1$ to 2. Since Simpson's formula gives the exact result, the area of the vertically shaded region must be the same as the area of the horizontally shaded region. Students found that an analogous thing happens for simultaneous plots over other intervals and other cubics; through exploration, they stumbled upon one of the great free lunches in mathematics: although Simpson's formula is designed only to integrate quadratics exactly, it in fact exactly integrates every cubic.

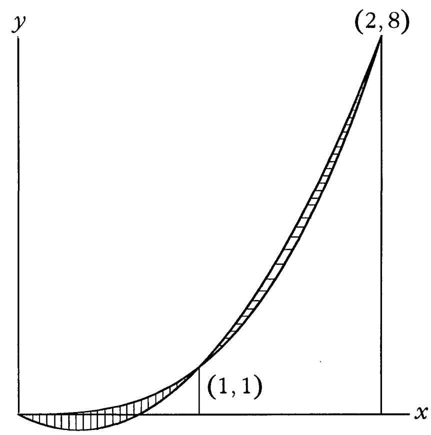

Figure 2

Maple made the proof of this easy. By direct integration, the area over $[a, c]$ and under the cubic

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

is

$$\frac{A}{4}(c^4 - a^4) + \frac{B}{3}(c^3 - a^3) + \frac{C}{2}(c^2 - a^2) + D(c - a).$$

Factoring out the base $(c - a)$ leaves

$$\frac{A}{4}(a^3 + a^2c + ac^2 + c^3) + \frac{B}{3}(a^2 + ac + c^2) + \frac{C}{2}(a + c) + D,$$

which turns out to be the Simpson average altitude $\frac{1}{6}[f(a) + 4f((a+c)/2) + f(c)]$. Maple easily proves this: applying the **simplify** command to their difference gives 0. The error in Simpson-approximating any analytic function therefore comes from terms of degree ≥ 4; the extra, "free" level of accuracy has made this method a perennial favorite.

This suggested taking another look at the rectangular and trapezoidal methods. Just as parabolic arcs are degree two approximations to a curve, the tops of rectangles and trapezoids are degree zero and one approximations. Do these likewise exactly integrate polynomials through one higher degree? For mid-point rectangles, the answer is yes: the curve is a line, and the errors to the right and to the left of the mid-point are the oppositely-signed areas of two congruent triangles, which cancel.

For trapezoids, the answer is clearly no, since the trapezoidal method always over estimates the area under any curve that is concave up, and under-estimates it for any curve that is concave down. We finally decided to see if the trapezoidal method could be improved.

Let us consider the parabola $y = x^2$ between $a = -1$ and $c = 1$, shown in Figure 3. The approximating trapezoid is a rectangle with upper vertices $(-1, 1)$ and $(1, 1)$. Its area is 2, far larger than the area under the parabola, which is only $\frac{2}{3}$. Figure 3, however, suggests an idea: move the altitudes closer together until the shaded areas cancel out. When does this occur?

In Figure 3, the base of the trapezoid is 2. The parabola intersects the top of the trapezoid at (d, d^2), so the altitude is d^2, giving an area of $2d^2$. If we want this to be equal to the area under the parabola, $\frac{2}{3}$, we take $d = \pm 1/\sqrt{3}$. Thus, moving the bases of the altitudes so they're $\pm 1/\sqrt{3}$ from the center exactly integrates the quadratic in this case.

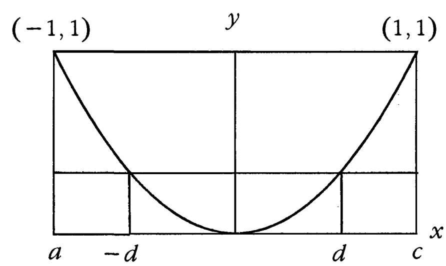

Figure 3

Remarkably, this works more generally. The exact area under $g(x) = Ax^2 + Bx + C$ from $x = a$ to $x = c$ is the same as that of the trapezoid whose base is $(c - a)$ and whose top is determined by altitudes based at points pulled in by a factor $1/\sqrt{3}$ towards the center $b = ((a + c)/2)$. This puts the altitude bases in our improved method at

$$\left(\frac{a+c}{2}\right) \pm \frac{1}{\sqrt{3}}\left(\frac{a-c}{2}\right)$$

rather than at the standard endpoints

$$\left(\frac{a+c}{2}\right) \pm 1\left(\frac{a-c}{2}\right)$$

(that is, a and c).

We can prove our claim much as we did before: first, direct integration shows that the area under the parabola is

$$\frac{A}{3}(c^3 - a^3) + \frac{B}{2}(c^2 - a^2) + C(c - a),$$

and this factors into the base $(c - a)$ times

$$\frac{A}{3}(a^2 + ac + c^2) + \frac{B}{2}(a + c) + C.$$

This last is the same as the average altitude of the trapezoid, which is

$$\frac{1}{2}\left[g\left(\frac{a+c}{2} + \frac{1}{\sqrt{3}}\left(\frac{a-c}{2}\right)\right) + g\left(\frac{a+c}{2} - \frac{1}{\sqrt{3}}\left(\frac{a-c}{2}\right)\right)\right].$$

Maple again easily verifies this—the **simplify** command shows that their difference is 0.

Amazingly, this gives us yet another free lunch: the improved trapezoid method is *so* much better that it actually integrates *all cubics* exactly! To see why, it's enough to show that it does this for $x^3 + g(x)$, with g as above. But since our new trapezoidal method integrates g exactly, we need only check that it does the same for x^3, giving $\frac{1}{4}(c^4 - a^4)$. Factoring out the trapezoid's base $(c - a)$ from this leaves

$$\tfrac{1}{4}(a^3 + a^2c + ac^2 + c^3);$$

this in turn is the average height

$$\frac{1}{2}\left[\left(\frac{a+c}{2} + \frac{1}{\sqrt{3}}\frac{a-c}{2}\right)^3 + \left(\frac{a+c}{2} - \frac{1}{\sqrt{3}}\frac{a-c}{2}\right)^3\right],$$

as Maple's **simplify** once again shows.

How do these various integration methods compare in accuracy? We know that for most problems, using midpoint rectangles is the least accurate, using trapezoids is next, and Simpson's formula is best. *However, the improved trapezoid method turns out to be even better than Simpson's formula*! It somehow doesn't seem right that a first-order method should be better than a second-order one. One attentive student asked, "Since the trapezoid method can be improved so it exactly integrates polynomials through degree three, can Simpson's method be improved so it does the same through degree four?" This question turned out to be the right one, and the approach used before of symmetrically moving the altitude base points made this look plausible, at least for $y = x^4$ (see Figure 4). The altitudes intersect the graph of $y = x^4$ in two points, and as these two points approach each other, the parabola passing through them and $(0, 0)$ flattens out, creating areas above and below $y = x^4$. Of course we'd like to get that particular parabola where the error-areas above and below $y = x^4$ cancel out. Let the parabola be $y = d^2x^2$. Over the interval $[-1, 1]$, the area under $y = d^2x^2$ is $2d^2/3$, and the area under $y = x^4$ is $\frac{2}{5}$. If these areas are equal, then $d = \sqrt{\frac{3}{5}}$. We might guess that, more generally, the outer altitudes in

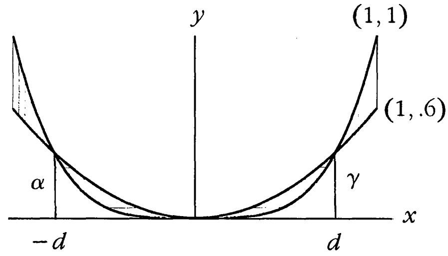

Figure 4

an improved Simpson formula ought to be based at

$$\left(\frac{a+c}{2}\right) \pm \sqrt{\frac{3}{5}}\left(\frac{a-c}{2}\right).$$

With these altitudes, what would the improved Simpson's formula be? The argument is a nice classroom review of the one used to get the usual formula: substituting the coordinates of any single point into the second-order equation $y = Ax^2 + Bx + C$ gives one linear equation in the variables A, B, C, so choosing three distinct points on this parabola gives a system of three linear equations. Solving for A, B, and C then expresses these coefficients in terms of the three chosen points, and one can substitute these into $\int_a^c (Ax^2 + Bx + C)\,dx$ to get the area as Base × (Average altitude). If the three points lie on the vertical lines through the two endpoints and the midpoint of $[a, c]$, we get the usual expression $(1\alpha + 4\beta + 1\gamma)/(1 + 4 + 1)$ for average altitude. However, if the vertical lines through the endpoints are symmetrically moved in by our factor of $\sqrt{\frac{3}{5}}$ (we continue to call these altitudes α, β, and γ), then the average altitude is

$$\frac{5\alpha + 8\beta + 5\gamma}{5 + 8 + 5}.$$

These average altitudes are always numerically identical. Example: if we take the parabola $y = x^2$ over $[-1, 1]$, then the altitudes taken at $x = -1, 0, 1$ are $\alpha = 1$, $\beta = 0$, $\gamma = 1$, and the usual Simpson's formula gives an average altitude of $((1 \cdot 1 + 4 \cdot 0 + 1 \cdot 1)/6) = \frac{1}{3}$. If, however, the altitudes are chosen at $x = -\sqrt{\frac{3}{5}}$, 0, $\sqrt{\frac{3}{5}}$, then $\alpha = \frac{3}{5}$, $\beta = 0$, $\gamma = \frac{3}{5}$, and we get $((5 \cdot \frac{3}{5} + 8 \cdot 0 + 5 \cdot \frac{3}{5})/18 = \frac{1}{3}$. The improved approximation to $\int_a^c f(x)dx$ is then $(c - a)((5\alpha + 8\beta + 5\gamma)/18)$, where

$$\alpha = f\left(\left(\frac{a+c}{2}\right) - \sqrt{\frac{3}{5}}\left(\frac{a-c}{2}\right)\right);\ \beta = f\left(\frac{a+c}{2}\right);\ \gamma = f\left(\left(\frac{a+c}{2}\right) + \sqrt{\frac{3}{5}}\left(\frac{a-c}{2}\right)\right).$$

It turns out that this improvement not only integrates all polynomials through degree 4, it even does it through degree 5!. (Once again, let Maple show that the appropriate difference is 0.)

How good are these improvements? For most differentiable functions, the improved version of Simpson's method will add at least half again the number of decimal places accuracy obtained from the usual Simpson version. Compared to the usual trapezoid method, improved Simpson's at least *triples* the number of accurate decimal places.

It turned out that these were not new discoveries. We'd unexpectedly stumbled upon two old results: the improved trapezoid and Simpson methods are actually cases $n = 2$ and 3 of Gaussian n-point quadrature. (See **[2]**, for example.) The zeros of the Legendre polynomial of degree n lead to an n-point quadrature formula that exactly integrates polynomials through degree $2n-1$. Our factors $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{3}{5}}$ are zeros of Legendre polynomials of order 2 and 3; correspondingly, our formulas work through degree 3 and 5. Tabulated solutions for $2 \le n \le 20$ appear in **[1]** and, through $n = 200$, in **[3]**. Of course, this approach does not fit into a typical beginning course.

Our experience showed that substantial improvements to the trapezoidal and Simpson's methods can be successfully introduced into a course for beginners, using little more than pictures and Maple's **simplify** command.

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Multiplying and Dividing Polynomials Using Geloxia

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One popular method for multiplying numbers during the Renaissance was that of "geloxia" or the grating **[1**, p. 209**]**. In this system the two numbers to be multiplied were written in an "L" shape above a grid of squares divided by diagonals. In Figure 1 is shown the multiplication of 2375 by 127 to give the product 301625. The entry in each square is the product of the two numbers at the top of the column and the

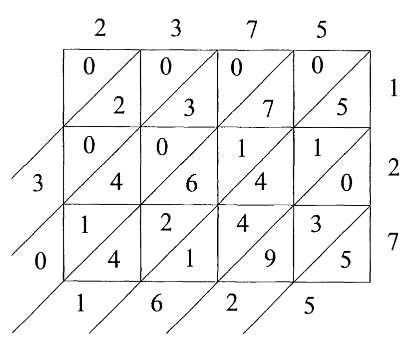

Figure 1