CLASSROOM CAPSULES

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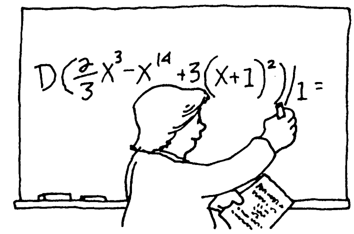

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia, Department of Mathematics, Santa Fe Community College Gainesville, FL 32606-6200

Maclaurin Expansion of Arctan *x* via L'Hôspital's Rule

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In [1], Spiegel used L'Hôspital's rule to compute the first three nonzero coefficients in the Maclaurin expansion of $\sin x$. He then judged this technique to be tedious and suggested its use only to motivate the conventional method using derivatives: students would then be more appreciative of the conventional method because it is "much less tedious." This is indeed the case for $\sin x$, but it is not necessarily so for all elementary functions. For instance, most calculus books do not directly obtain the Maclaurin series for $f(x) = \text{Arctan}\, x$, undoubtedly because the nth derivative of f is not straightforward. Instead, most authors start with the Maclaurin expansion for $1/(1+t)$, then replace t with t^2 and integrate the resulting series termwise from 0 to x (where $-1 < x < 1$) to arrive at the desired result. This indirect approach uses the fact that a power series is uniformly convergent on compact subsets of the interval of convergence. The purpose of this note is to obtain the Maclaurin expansion of Arctan x by using L'Hôspital's rule instead of using uniform convergence of series or successive derivatives of Arctan x.

Assume that

$$f(x) = \text{Arctan}\, x = \sum_{n=0}^{\infty} c_n x^n \tag{1}$$

is an identity in x. One needs to determine the c_n's. First, since f is an odd function, we readily get from (1) that

$$-\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n c_n x^n.$$

For this to be an identity in x, it is necessary that $-c_{2i} = c_{2i}$ for $i = 0, 1, 2, \ldots$.

Hence $c_{2i} = 0$ for all nonnegative integers i, and identity (1) becomes

$$\begin{aligned} \text{Arctan } x &= c_1 x + c_3 x^3 + \cdots + c_{2n+1} x^{2n+1} + \cdots \\ &= c_1 x + O(x^3). \end{aligned} \tag{2}$$

To get c_1, first divide both sides of (2) by x, isolating c_1:

$$\frac{\text{Arctan } x}{x} = c_1 + O(x^2).$$

Hence

$$c_1 = \lim_{x \to 0} \frac{\text{Arctan } x}{x}$$

and so L'Hôspital's rule gives

$$c_1 = \lim_{x \to 0} \frac{1}{1 + x^2} = 1.$$

Next get c_3. We have

$$\text{Arctan } x = x + c_3 x^3 + O(x^5),$$

which is equivalent to

$$\frac{\text{Arctan } x - x}{x^3} = c_3 + O(x^2).$$

So, as before,

$$\begin{aligned} c_3 &= \lim_{x \to 0} \frac{\text{Arctan } x - x}{x^3} \\ &= \lim_{x \to 0} \frac{\dfrac{1}{1 + x^2} - 1}{3x^2} \\ &= \lim_{x \to 0} \frac{-1}{3(1 + x^2)} \\ &= -\frac{1}{3}. \end{aligned}$$

Hence

$$\text{Arctan } x = x - \tfrac{1}{3} x^3 + O(x^5).$$

Continuing this process gives, for $n \geq 1$,

$$c_{2n+1} = \lim_{x \to 0} \frac{\operatorname{Arctan} x - x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \cdots + \frac{(-1)^n}{2n-1}x^{2n-1}}{x^{2n+1}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x^2} - 1 + x^2 - x^4 + \cdots + (-1)^n x^{2n-2}}{(2n+1)x^{2n}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x^2} + \frac{-1 + (-1)^n x^{2n}}{1+x^2}}{(2n+1)x^{2n}}$$

$$= \lim_{x \to 0} \frac{(-1)^n x^{2n}}{(2n+1)x^{2n}(1+x^2)}$$

$$= \lim_{x \to 0} \frac{(-1)^n}{(2n+1)(1+x^2)}$$

$$= \frac{(-1)^n}{2n+1}.$$

Therefore, we conclude that

$$\operatorname{Arctan} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

It is easy to show that the interval of convergence for this series is $[-1, 1]$.

The above techniques can be used to find the Taylor series expansion for an arbitrary (real) analytic function. However, for convenience, take $x_0 = 0$ and assume that f has a power series representation of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then $f(0) = c_0$ and so

$$f(x) = f(0) + c_1 x + O(x^2).$$

Hence,

$$\frac{f(x) - f(0)}{x} = c_1 + O(x)$$

and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = c_1.$$

So,

$$f(x) = f(0) + f'(0)x + c_2 x^2 + O(x^3),$$

or

$$\frac{f(x)-f(0)-f'(0)x}{x^2}=c_2+O(x).$$

Therefore,

$$c_2=\lim_{x\to 0}\frac{f(x)-f(0)-f'(0)x}{x^2}.$$

Using L'Hôspital's rule on the left-hand side of the above identity yields

$$c_2=\lim_{x\to 0}\frac{f'(x)-f'(0)}{2x}$$

which is equivalent to

$$c_2=\tfrac{1}{2}f''(0).$$

Continuing in this fashion gives $c_n=(1/n!)f^{(n)}(0)$ for $n=0,1,2,3,\dots$.

For most functions encountered in elementary calculus, there are easier ways to find Maclaurin expansions. The technique of using L'Hôspital's rule can be viewed as supplementing, rather than replacing, the standard techniques.

Reference

1. M. R. Spiegel, L'Hôspital's rule and expansion of functions in power series, *American Mathematical Monthly* 62 (1955) 358–360, reprinted in T. M. Apostol *et al.*, *Selected Papers in Calculus*, Mathematical Association of America, Washington, DC, 1969, 211–213.

An Exponential Rule

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Students of elementary calculus who are quite comfortable with the power rule

$$\frac{d}{dx}[f(x)]^p=p[f(x)]^{p-1}f'(x)$$

become discouraged to discover that it fails when $(d/dx)p^{g(x)}=p^{g(x)}g'(x)\ln p$, $p>0$. They become further chagrined to see that a derivative of the form $(d/dx)f(x)^{g(x)}$ requires a different technique. They ask why, when these are structurally similar, are they so different? Insightful students will note that the above constant p may be considered a constant function, so some connection must exist. To reduce the mystery and prompt discussion, one might introduce a general exponential rule

$$\frac{d}{dx}f(x)^{g(x)}=g(x)[f(x)]^{g(x)-1}f'(x)+f(x)^{g(x)}g'(x)\ln f(x),\qquad f(x)>0$$

Ostrowski, *Differential and Integral Calculus*, Scott, Foresman, 1968, p. 276].

The first term appears as the power rule with $g(x)$ treated as a constant and the second term the rule for functional exponents with $f(x)$ treated as a constant.