Moments on a Rose Petal

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For several years, I have been insisting that my first-year calculus students check every nontrivial integration by differentiation. This teaching strategy, based on the pious hope that students will acquire the habit of finding and eliminating readily detectable errors, is occasionally at cross-purposes with the integration strategies commonly suggested in textbooks.

The best examples are integrals of the form $\int f(mx)g(nx)\,dx$, where f, g are the sine or cosine functions and m, n are numbers, neither of which is an integer multiple of the other. Most texts (34 of 44 surveyed) recommend the use of the "prosthaphaeresis" rule [V. E. Thoren, Prosthaphaeresis revisited, *Historia Mathematica* 15 (1988) 32–39]:

$$\begin{aligned}\sin A \sin B &= [\cos(A-B) - \cos(A+B)]/2\\ \cos A \cos B &= [\cos(A+B) + \cos(A-B)]/2\\ \sin A \cos B &= [\sin(A+B) + \sin(A-B)]/2.\end{aligned}\tag{1}$$

While conceptually simple if one can remember the rule, the technique yields forms of the integral that are extremely difficult to check. Just watch a student's eyes glaze when confronted with maneuvers like

$$\cos x = \cos(5x - 4x) = \cos 5x \cos 4x + \sin 5x \sin 4x.$$

The minority report [M. R. Embry, J. F. Schell, and J. P. Thomas, *Calculus and Linear Algebra: An Integrated Approach*, Saunders, Philadelphia, 1972 and A. Shenk, *Calculus and Analytic Geometry*, Scott, Foresman, Glenview, IL, 1978] advocates the use of integration by parts. This technique is somewhat longer but spares the student memorization of what are by now virtually single-purpose formulas and simplifies checking considerably by preserving the original arguments in the trigonometric functions. Omitting constants and assuming $m^2 - n^2 \neq 0$, the integrals obtained are

$$\begin{aligned}\int \sin mx \sin nx\,dx &= (m \cos mx \sin nx - n \sin mx \cos nx)/(n^2 - m^2)\\ \int \cos mx \cos nx\,dx &= (n \cos mx \sin nx - m \sin mx \cos nx)/(n^2 - m^2)\\ \int \cos mx \sin nx\,dx &= (n \cos mx \cos nx + m \sin mx \sin nx)/(m^2 - n^2).\end{aligned}\tag{2}$$

This recommendation underwent the severest test to date when, in the dying seconds of a lecture to a second-year intermediate calculus class, I challenged the students to compute the centroid of one "petal" of the 3-leaved rose $r = \sin 3\theta$. Several students successfully formulated

$$M_x = \int\int y\,dA = \int_0^{\pi/3}\int_0^{\sin 3\theta} r^2 \sin\theta\,dr\,d\theta = (1/3)\int_0^{\pi/3} \sin\theta \sin^3 3\theta\,d\theta,$$

and M_y analogously, but none was able to find the required indefinite integral.

Integration by parts being initially unappealing, prosthaphaeresis was given two chances. Three applications of (1) gave the remarkable formulation

$$\sin\theta \sin^3 3\theta = \frac{1}{8}(3\cos 2\theta - 3\cos 4\theta - \cos 8\theta + \cos 10\theta),$$

but a check by differentiation would be daunting indeed. Writing

$$\begin{aligned}\sin\theta \sin^3 3\theta &= (\sin\theta \sin 3\theta)\sin^2 3\theta \\ &= \frac{1}{2}[\cos 2\theta - \cos 4\theta](1-\cos 6\theta)/2\end{aligned}$$

seems more promising, but merely leads more quickly to the same identity. The reader is invited to explore the delights of substituting $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, expanding the cube, and hacking through the resulting jungle of double angle formulas.

Head unbowed though sufficiently bloody to keep Macbeth from his banquet, I tried the time-honored technique of generalizing the problem. Let n be any positive integer no smaller than 2, and consider $I = \int \sin\theta \sin^n m\theta\, d\theta$. While it is feasible to integrate the factor $\sin^3 3\theta$ from the original example, it is entirely unfeasible to integrate the more general term $\sin^n m\theta$ so we are drawn inexorably to choose $u = \sin^n m\theta$, $dv = \sin\theta\, d\theta$.

Two applications of the parts formula and one of $\cos^2 m\theta = 1 - \sin^2 m\theta$ yields the following reduction formula:

$$\begin{aligned}(1-m^2n^2)I = &-\cos\theta \sin^n m\theta + mn \sin^{n-1} m\theta \cos m\theta \sin\theta \\ &- m^2 n(n-1)\int \sin^{n-2} m\theta \sin\theta\, d\theta. \qquad (3)\end{aligned}$$

Applying (3) to our original integral, we obtain, with the aid of (2):

$$\begin{aligned}-80I &= G(\theta) \\ &= -\cos\theta \sin^3 3\theta + 9\sin^2 3\theta \cos 3\theta \sin\theta - \frac{27}{4}(\cos\theta \sin 3\theta - 3\sin\theta \cos 3\theta).\end{aligned}$$

(The reader is invited to check by differentiation.)
The moment M_x of the petal with respect to the horizontal axis is then $-[G(\pi/3) - G(0)]/240 = 27\sqrt{3}/640$. Hence,

$$\bar{y} = M_x/(\text{Area}) = 81\sqrt{3}/160\pi, \text{ and by symmetry, } \bar{x} = \bar{y}\cot(\pi/6) = 243/160\pi.$$

Similarly, one can obtain the analogous reduction formula for $J = \int \cos\theta \sin^n m\theta\, d\theta$, which is given by

$$\begin{aligned}(1-m^2n^2)J = &\sin\theta \sin^n m\theta + mn \sin^{n-1} m\theta \cos m\theta \cos\theta \\ &+ m^2 n(n-1)\int \cos\theta \sin^{n-2} m\theta\, d\theta.\end{aligned}$$

This formula also assists in computing second moments. For instance, returning to the example of $r = \sin 3\theta$, we find

$$I_x = \int\int y^2\,dA = \int_0^{\pi/3}\int_0^{\sin 3\theta} r^3 \sin^2\theta\,dr\,d\theta$$

$$= (1/4)\int_0^{\pi/3} \sin^2\theta \sin^4 3\theta\,d\theta$$

$$= (1/8)\int_0^{\pi/3} (1-\cos 2\theta)\sin^4 3\theta\,d\theta$$

$$= (1/8)\int_0^{\pi/3} \sin^4 3\theta\,d\theta - (1/16)\int_0^{2\pi/3} \cos\alpha \sin^4(3\alpha/2)\,d\alpha,$$

where $\alpha = 2\theta$.

Note: Shenk's fourth edition, published by Scott, Foresman in 1988, has fallen from grace by switching to the prosthaphaeresis method for products of sines and cosines with dissimilar arguments.

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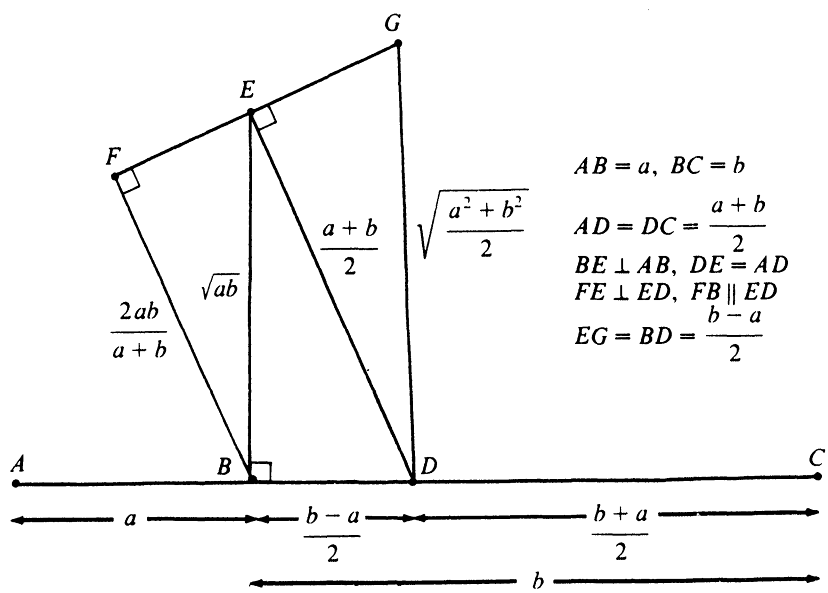
Harmonic, Geometric, Arithmetic, Root Mean Inequality



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