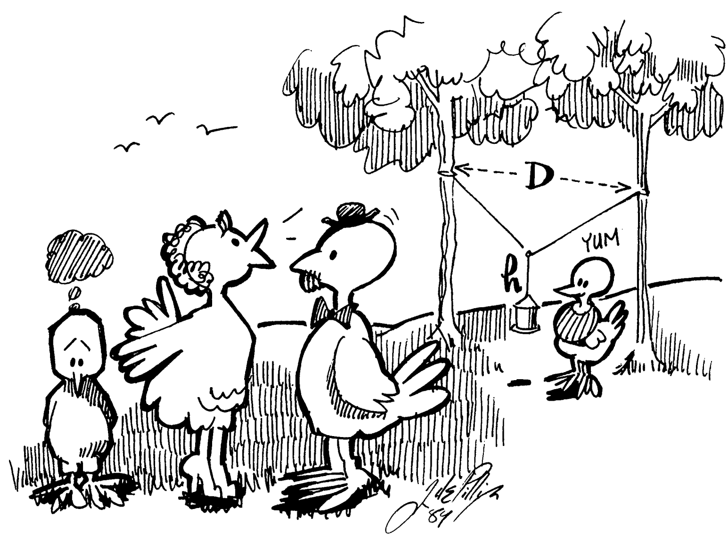
configuration is best. On the other hand, $L_V < L_T$ whenever $D > \frac{3}{2}d$, so if $D > 2\sqrt{3}\,d$, the V configuration is best.

It is instructive to have students draw the optimal configurations to scale for several different values of the parameters. Better yet, to obtain a physical solution to the minimization problem, wedge three thin pegs between two transparent plates and then dip the apparatus into soapy water. The film makes a configuration that minimizes the total distance connecting the three pegs [Richard Courant and Herbert Robbins, *What is Mathematics*?, Oxford University Press, New York, 1941, p. 392]. By projecting the image of the soap film onto a screen with the aid of an overhead projector, students may then notice, as Steiner did, that the angles between the pegs in the Y configuration are equal. Having made that observation, the students can verify it by computing

$$\frac{D/2}{d-h} = \sqrt{3} = \tan 60^\circ.$$

For an overview of Steiner's problem, see the article by H. W. Kuhn in G. B. Dantzig and B. C. Eaves, *Studies in Optimization*, MAA, Washington, D.C., 1974.



"I don't understand it. Rodney feels he can't use the feeder since he flunked calculus!"

Determinants of Sums

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A determinant formula. In an issue of the *College Mathematics Journal* [Evaluating "uniformly filled" determinants, *CMJ* 19 (1988) 343–345], S. M. Goberstein exhibits a formula for computing the determinant of a matrix V obtained from a matrix U by adding the scalar v to every entry of U. The author then evaluates several determinants of this type and mentions that freshmen mathematics majors in the Soviet Union have used the method for several decades.

The method that Goberstein exhibits for determinants of this type is a very special case of an interesting formula for the determinant of the sum of any two matrices. This formula for $\det(A+B)$ can be easily derived from the Laplace expansion theorem, and consequently is readily accessible to students in an elementary linear algebra course. As we shall see, the formula can also be used to directly obtain important results about the characteristic polynomial, and about relationships between the characteristic roots and the subdeterminants of A. The formula in question is one of those "folk" results whose precise origins are difficult to trace. It appears in [Marvin Marcus, *Finite Dimensional Multilinear Algebra*, *Part II*, Marcel Dekker Inc., New York, 1975, pp. 162–163] with a related result about the permanent. In fact, the permanent version of the formula provides a simple proof of the classical formula that counts the number of derangements of an n element set [Herbert J. Ryser, *Combinatorial Mathematics*, The Mathematical Association of America, Carus Mathematical Monograph #14, 1963, pp. 22–28]. However, I first learned the determinant version from Professor Emilie Haynsworth more than twenty years ago. The formula is

$$\det(A+B)=\sum_{r}\sum_{\alpha,\beta}(-1)^{s(\alpha)+s(\beta)}\det(A[\alpha|\beta])\det(B(\alpha|\beta)). \tag{1}$$

In (1): A and B are n-square matrices; the outer sum on r is over the integers $0,\dots,n$; for a particular r, the inner sum is over all strictly increasing integer sequences α and β of length r chosen from $1,\dots,n$; $A[\alpha|\beta]$ (square brackets) is the r-square submatrix of A lying in rows α and columns β; $B(\alpha|\beta)$ is the $(n-r)$-square submatrix of B lying in rows complementary to α and columns complementary to β; and $s(\alpha)$ is the sum of the integers in α. Of course, when $r=0$ the summand is taken to mean $\det(B)$ and when $r=n$, it is $\det(A)$.

The proof of (1) is a very simple consequence of the linearity of the determinant in each row of the matrix, and the standard Laplace expansion theorem. Here are the details of the argument. Write

$$\det(A+B)=\det\left(A_{(1)}+B_{(1)},\dots,A_{(n)}+B_{(n)}\right) \tag{2}$$

where $A_{(i)}$ denotes the ith row of A. The right side of (2) formally acts just like a product of the "binomials" $A_{(i)}+B_{(i)}$, $i=1,\dots,n$: this is the meaning of det being linear in the rows. Thus for each r chosen from $0,\dots,n$, the right side of (2) contributes a sum over all α of length r of terms of the form

$$\det\left(B_{(\alpha'_1)},\dots,A_{(\alpha_1)},\dots,A_{(\alpha_r)},\dots,B_{(\alpha'_{n-r})}\right) \tag{3}$$

in which $A_{(\alpha_t)}$ occupies row position α_t in (3), $t=1,\dots,r$ and $B_{(\alpha'_t)}$ occupies row position α'_t, $t=1,\dots,n-r$. The sequence α' is strictly increasing and complementary to α in $1,\dots,n$. Let X_α denote the n-square matrix in (3), i.e., $A_{(\alpha_t)}$ is row α_t of X_α, $t=1,\dots,r$, and $B_{(\alpha'_t)}$ is row α'_t of X_α, $t=1,\dots,n-r$. (In the cases $r=0$ and $r=n$, X_α is B and A, respectively). In terms of X_α we have

$$\det(A+B)=\sum_{r}\sum_{\alpha}\det(X_\alpha). \tag{4}$$

Use Laplace expansion on rows numbered α in X_α to obtain

$$\det(X_\alpha)=(-1)^{s(\alpha)}\sum_{\beta}(-1)^{s(\beta)}\det(X_\alpha[\alpha|\beta])\det(X_\alpha(\alpha|\beta)). \tag{5}$$

But according to the definition of X_α,

$$X_\alpha[\alpha|\beta] = A[\alpha|\beta], \tag{6}$$

and

$$\begin{aligned} X_\alpha(\alpha|\beta) &= X_\alpha[\alpha'|\beta'] \\ &= B[\alpha'|\beta'] \\ &= B(\alpha|\beta). \end{aligned} \tag{7}$$

Substitute (6) and (7) in (5) and then replace $\det(X_\alpha)$ in (4) by (5). The result is formula (1).

Some examples. The examples in Goberstein's article are all of the form $A + B$ in which B has rank 1. Any rank 1 matrix is of the form $B = uv^T$ where u and v are nonzero $n \times 1$ matrices. Every subdeterminant of uv^T of size 2 or more is 0 so that the only summands that survive on the right side of formula (1) are those corresponding to $r = n$ and $r = n - 1$. For $r = n$ the single summand is $\det(A)$; for $r = n - 1$ the interior sum may be rewritten as

$$\sum_{i,j=1}^{n} (-1)^{i+j}\det(A(i|j))u_i v_j. \tag{8}$$

The i, j entry of $\operatorname{adj}(A)$, the adjugate of A (sometimes called the adjoint of A), is $(-1)^{i+j}\det A(j|i)$ and hence (8) can be written as $v^T\operatorname{adj}(A)u$. Thus we have the rather neat formula

$$\det(A + uv^T) = \det(A) + v^T\operatorname{adj}(A)u. \tag{9}$$

The first matrix considered in Goberstein's article is

$$\operatorname{diag}(1-n, 2-n, 3-n, \ldots, 0) + nJ \tag{10}$$

in which J is the n-square matrix all of whose entries are 1. Let A be the diagonal matrix in (10) and let $u = ne$, $v = e$ where e is the $n \times 1$ matrix all of whose entries are 1. Obviously $\det(A) = 0$ and the only term that survives in (8) corresponds to $i = j = n$, namely

$$n \cdot (1-n)(2-n) \cdots 1 = (-1)^{n-1}n!.$$

The second and third matrices considered in Goberstein's article are

$$-P + J, \tag{11}$$

and the $2n$-square matrix

$$M + J. \tag{12}$$

In (11), P is the matrix whose only nonzero entries are 1's on the sinister diagonal, i.e., $P_{i,n-i+1} = 1$, $i = 1, \ldots, n$; the matrix M in (12) is a direct sum of n copies of the 2-square matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and J is the $2n$-square matrix of 1's. In (11) note that $\det(P)=(-1)^p$ where $p = n \operatorname{div} 2$, i.e., the largest integer in $n/2$. Since $P^2 = I_n$ we have $P^{-1}=P$,

$$\begin{aligned}\operatorname{adj}(P) &= \det(P)\cdot P^{-1}\\ &= (-1)^p P.\end{aligned}$$

Then

$$\begin{aligned}\operatorname{adj}(-P) &= \det(-P)(-P)^{-1}\\ &= (-1)^{n+1}\det(P)\cdot P\\ &= (-1)^{n+1}(-1)^p P\\ &= (-1)^{n+p+1}P,\end{aligned}$$

so that (9) becomes

$$\begin{aligned}\det(-P+J) &= \det(-P) + e^T\operatorname{adj}(-P)e\\ &= (-1)^{n+p} + (-1)^{n+p+1}e^TPe\\ &= (-1)^{n+p}(1-n).\end{aligned}$$

It is easy to check that $n+p$ and $n(n+1)/2$ have the same parity and thus

$$\det(-P+J) = (-1)^{n(n+1)/2}(1-n). \tag{13}$$

The value of $\det(M+J)$ is equally simple to compute. Observe that M also satisfies $M^{-1}=M$, and that $\det(M)=(-1)^n$. Thus, as was the case with the matrix P,

$$\operatorname{adj}(M) = (-1)^n M,$$

and again taking $u=v=e$, (9) specializes to

$$\begin{aligned}\det(M+J) &= \det(M) + (-1)^n e^TMe\\ &= (-1)^n + (-1)^n(2n)\\ &= (-1)^n(1+2n).\end{aligned} \tag{14}$$

The arguments used so far point to a unified formula incorporating (13) and (14). Let A be an arbitrary n-square real orthogonal matrix. Then

$$\begin{aligned}\operatorname{adj}(A) &= \det(A)A^{-1}\\ &= \det(A)A^T.\end{aligned}$$

If $u=ce$ and $v=e$ then $A+uv^T = A+cJ$ and (9) becomes

$$\begin{aligned}\det(A+cJ) &= \det(A) + \det(A)ce^TA^Te\\ &= \det(A)(1+ca)\end{aligned} \tag{15}$$

where a is the sum of the entries in A. A referee points out that the formula (15) holds for matrices that satisfy $e^TAe = e^TA^{-1}e$. The proof is identical.

One of the troika of excellent referees assigned to this paper suggests several additional examples on which to apply (9) or some variant of it. The first of these is the matrix

$$P + B \tag{16}$$

where B is the n-square rank 1 matrix whose kth row is

$$B_{(k)} = k[1\,2\,3 \ldots n], \qquad k = 1, \ldots, n,$$

and P is the matrix in (11). Note that $B = uv^T$ with

$$u = v = [1\,2\,3 \ldots n]^T.$$

Then (9) becomes

$$\begin{aligned}\det(P + B) &= \det(P) + (-1)^p u^T P u \\ &= (-1)^p \left(1 + \sum_{k=1}^{n} k(n - k + 1)\right).\end{aligned}$$

The summation in this last formula is quickly evaluated as

$$n(n+1)(n+2)/6$$

and thus

$$\det(P + B) = (-1)^p (1 + n(n+1)(n+2)/6), \qquad p = n \operatorname{div} 2. \tag{17}$$

As another example we will evaluate the function

$$f(x) = \det(I_n + B) \tag{18}$$

where B is the n-square rank 1 matrix whose kth row is

$$B_{(k)} = x^k [1 x \cdots x^{n-1}], \qquad k = 1, \ldots, n$$

(x is an indeterminate). Note that

$$B = xuu^T$$

where

$$u = [1\,x\,x^2 \cdots x^{n-1}]^T.$$

Again applying (9) we have

$$\begin{aligned} f(x) &= 1 + xu^T u \\ &= 1 + x \sum_{k=0}^{n-1} x^{2k} \\ &= 1 + \sum_{k=0}^{n-1} x^{2k+1}. \end{aligned}$$

The general idea of how to use (9) should be clear, so we leave a final example as an exercise for the reader: evaluate

$$f(x) = \det(D + xJ)$$

where $D = \text{diag}(1, 2, \ldots, n)$. The answer is

$$f(x) = n!\left(1 + x \sum_{k=1}^{n} \frac{1}{k}\right).$$

The characteristic polynomial. The formula (1) can be directly applied to the characteristic matrix $\lambda I_n - B$ by replacing A by λI_n and B by $-B$. Since $\det(I_n[\alpha|\beta])$ is 0 for $\alpha \neq \beta$ and is 1 for $\alpha = \beta$, we have

$$\begin{aligned}\det(\lambda I_n - B) &= \sum_{r=0}^{n} \sum_{\alpha} \lambda^r \det(-B(\alpha|\alpha)) \\ &= \sum_{r=0}^{n} \lambda^r (-1)^{n-r} b_{n-r} \qquad (19)\end{aligned}$$

where b_{n-r} is the sum of all $(n-r)$-square principal subdeterminants of B. If $\lambda_1, \ldots, \lambda_n$ are the characteristic roots of B then

$$\begin{aligned}\det(\lambda I_n - B) &= \prod_{i=1}^{n} (\lambda - \lambda_i) \\ &= \sum_{r=0}^{n} \lambda^r (-1)^{n-r} e_{n-r} \qquad (20)\end{aligned}$$

where e_k is the kth elementary symmetric polynomial in $\lambda_1, \ldots, \lambda_n$. Matching coefficients in (19) and (20) we have

$$e_k = b_k, \qquad k = 1, \ldots, n.$$

Of course, $k = 1$ and $k = n$ are the familiar

$$\text{tr}(B) = \sum_{i=1}^{n} \lambda_i,$$

and

$$\det(B) = \prod_{i=1}^{n} \lambda_i.$$

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On 'Uniformly Filled' Determinants

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Let U be a square matrix of order n, and let v be any number. Let $V = (u_{ij} + v)_{i,j=1}^{n}$ be the matrix obtained from U by adding v to each entry of U. In the classroom capsule [**1**] it was observed that

$$\det V = \det U + v \cdot \sum_{i,j=1}^{n} Cof(U)_{ij}, \qquad (1)$$