References

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Linear Algebra in the Financial World

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In the beginning of 1999 some financial funds bought call options on gold with an exercise price of \$390 an ounce and exercise date December 1999. This means that they had the option to buy gold at \$390 an ounce in December. The reason for entering into the contract was the expectation (based on the millennium fever and Y2K scare) that the price of gold would rise dramatically in December 1999. For example, if the price did rise to \$400 an ounce, the pay-off would be \$10 an ounce. The price of each call option was \$1 per ounce, so the profit on 10,000 call options, would have been 10,000 × (\$400 − \$390) − 10,000(\$1) or \$90,000! If the price remained below \$390, the option would not be exercised and \$10,000 lost. How was the price of \$1 per call option determined so that the contract was fair to both parties? Can all possible contracts be priced fairly? These important problems involve concepts like arbitrage and complete markets, and this is what I want to discuss with the aid of linear algebra pictures as in [**3**].

Some Financial Concepts

Consider a discrete time single period model of a market where we have N assets. Their prices are given by numbers $S_1^0, S_2^0, \ldots, S_N^0$ at the initial time $t = 0$ and their prices at some future time $t = 1$ are random variables $S_1^1, S_2^1, \ldots, S_N^1$. If we allow *arbitrage* in a model of a market in equilibrium, then it means you could start with no money and have a chance of ending up with some money with no risk at all. Since this is not a reasonable situation, we assume there are no arbitrage opportunities. It can be proved [**2**] that this is equivalent to the existence of a risk neutral probability measure Q. What do we mean by a risk neutral probability measure? Well, if there are K possible states of the world $\omega_1, \ldots, \omega_K$, that can be realized at a certain time in the future, then $Q = (Q(\omega_1), \ldots, Q(\omega_K))$ where $Q(\omega_i) > 0$ for $i = 1, \ldots, K$ and $\sum_{i=1}^{K} Q(\omega_i) = 1$. Furthermore, the expectation with respect to Q of the future asset prices S_j^1, is $(1 + r)S_j^0$ where r is the interest rate. Roughly speaking, Q is that measure which ensures that *on average* you can do no better with asset j than you would have done by initially depositing amount S_j^0 in the bank. (In a specific state of the world you may of course do better). This Q ensures that claims or contracts on these N assets are priced fairly for both parties entering into the contract. Mathematically:

$$E_Q[Sj] = \sum_{k=1}^{K} Q(\omega_k) S_j^1(\omega_k) = (1 + r)S_j^0, \qquad \text{for each } j = 1, \ldots, N$$

and, in linear algebra terms, Q is that positive vector satisfying

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ S_1(\omega_1) & S_1(\omega_2) & \cdots & S_1(\omega_K) \\ & & \cdots & \\ S_N(\omega_1) & S_N(\omega_2) & \cdots & S_N(\omega_K) \end{pmatrix} \begin{pmatrix} Q(\omega_1) \\ Q(\omega_2) \\ \cdots \\ Q(\omega_K) \end{pmatrix} = \begin{pmatrix} 1 \\ (1+r)S_1^0 \\ \cdots \\ (1+r)S_N^0 \end{pmatrix}$$

or $BQ = \hat{S}$. Since we assume an arbitrage-free world, there is either a unique solution Q or infinitely many solutions Q. In the former case, the nullity of B is 0. In the latter case, we know the solution set is $Q_p + N(B)$ where Q_p is a particular solution and $N(B)$ is the null space of B. Moreover,

$$N(B) = \{Q - Q' : Q \text{ and } Q' \text{are solutions of } BQ = \hat{S}\}. \tag{1}$$

We now briefly discuss the very important pricing of contingent claims, such as options. (See also [**1**]). A *contingent claim* $X = (X(\omega_1), X(\omega_2), \ldots, X(\omega_K))$ can be seen as a payoff, say from a call option. We want such a claim to be *attainable*. This means that there exists a trading strategy $(H_0, H_1, \ldots H_N)$ such that the value of the portfolio at the end of the specified time equals X. That is,

$$H_0 + \sum_n H_n S_n(\omega_j) = X(\omega_j) \quad \text{for each } j = 1, \ldots K,$$

where H_0 is the number of dollars invested in the bank and H_n the number of units of asset n that are bought or sold.

In linear algebra terms, claim X is attainable iff

$$\begin{pmatrix} 1 & S_1(\omega_1) & \cdots & S_N(\omega_1) \\ 1 & S_1(\omega_2) & \cdots & S_N(\omega_2) \\ & & \cdots & \\ 1 & S_1(\omega_k) & \cdots & S_N(\omega_k) \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ \cdots \\ H_N \end{pmatrix} = \begin{pmatrix} X(\omega_1) \\ X(\omega_2) \\ \cdots \\ X(\omega_k) \end{pmatrix}$$

or $AH = X$ has a solution H. Thus,

$$X \text{ is attainable if and only if } X \in C(A). \tag{2}$$

Note that if rank $(A) = K$ then every claim X is attainable. In this case we say the market is *complete*.

Two Results Made Easy

In [**2**] we have the following two important results:

1. The market is complete iff there is only one risk neutral probability measure Q.
2. Claim X is attainable iff $E_Q[X/(1+r)]$ takes the same value for every possible Q.

The proofs can be considerably simplified using the linear algebra insights above. First look at result 1.

> The market is complete $\Leftrightarrow$ rank $(A) = K$ $\Leftrightarrow$ rank $(A^T) = K$ $\Leftrightarrow$ nullity $(A^T) = K - K$ $\Leftrightarrow$ nullity $(B) = 0$ $\Leftrightarrow$ $BQ = \hat{S}$ has a unique solution Q.

Now result 2: In a complete market every claim X is attainable (by definition) and there is only one Q. We thus need prove 2 only for X in an incomplete market. Since we are assuming no arbitrage, this means there are infinitely many solutions to $BQ = S$. Let $E_Q[X/(1+r)] = E_{Q'}[X/(1+r)]$ for any risk neutral Q, Q'. Then

$$\sum Q(\omega_i)X(\omega_i) = \sum Q'(\omega_i)X(\omega_i); \quad \text{i.e.} \quad Q \cdot X = Q' \cdot X \text{ for any } Q, Q'.$$

But $X \cdot (Q - Q') = 0$ for any Q, Q' means $X \in N(B)^\perp$ (by (1)). Since $(N(B))^\perp = C(B^T) = C(A)$ ($C(\cdot)$ the column space; see [**3**]), we have that X is attainable, by (2).

Conversely, if X is attainable, it is straightforward to prove that $E_Q[X/(1+r)]$ takes the same value for all risk neutral Q. In fact, if X is attainable then $X \in C(A) = C(B^T)$ so that X is a linear combination of the rows of B. We can write $X = YB$ for some row vector Y. Now, $E_Q[X/(1+r)] = \sum Q(\omega_i)X(\omega_i) = XQ = YBQ = Y\hat{S}$, a result that is independent of Q. The value $E_Q[X/(1+r)]$ then gives the fair price of the claim X.

It seems that important results in discrete time models can be proved quite elegantly if one "remembers" some simple linear algebra pictures. One can also show students that facts like $N(A)^\perp = C(A^T)$ can have useful applications.

Exercise: Let $K = 2$, $N = 1$, $r = 0.1$, $S^0 = 5$, $S(\omega_1) = 22/5$, $S(\omega_2) = 33/5$. That is, we have two states of the world, an interest rate of 10% for our bank account and one risky asset.

a. Show that there is a unique risk neutral probability measure Q.
b. Find a trading strategy which generates claim $X = (2, 0)$. X could be an option.
c. Find the price of the claim.

Such problems can also be discussed in terms of column spaces, null spaces, etc.

References

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Erratum

In the November issue (p. 364) was the alleged answer to a triangle-counting problem. It was wrong. Instead of 50 triangles, Donald Taranto (donaldh@saclink.csus.edu) and Richard Syverson (rsyverson@mediaone.net) both counted 56. This is the final result; there will be no recount.

