

## The Series for $e$ via Integration

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This note describes yet another way of obtaining the classical series expansion for  $e$ , namely

$$e = \lim_{n \rightarrow \infty} \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\}$$

by means of integrals, *not* by Taylor polynomials. This derivation is an interesting application of integration by parts and it offers a nice introduction to infinite series.

Let

$$x_n = \int_0^1 t^n e^{-t} dt$$

for  $n = 0, 1, 2, \dots$ . One may easily show that

$$x_0 = -\frac{1}{e} + 1$$

and integrating by parts yields

$$x_n = -\frac{1}{e} + nx_{n-1}, \quad n \geq 1.$$

Repeated use of this formula gives

$$\begin{aligned} x_n &= -\frac{1}{e} + n \left( -\frac{1}{e} + (n-1)x_{n-2} \right) \\ &= \cdots \\ &= -\frac{1}{e} (1 + n + n(n-1) + \cdots + n!) + n! x_0 \\ &= -\frac{1}{e} (1 + n + n(n-1) + \cdots + n!) + n! \left( -\frac{1}{e} + 1 \right). \end{aligned}$$

This may be readily verified by induction. By considering the integrand in the definition of  $x_n$ , one sees that  $|x_n| \leq 1$  for all  $n$ . We then have

$$0 = \lim_{n \rightarrow \infty} \frac{x_n}{n!} = \lim_{n \rightarrow \infty} \left[ -\frac{1}{e} \left( \frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + \frac{1}{1!} + 1 \right) + 1 \right]$$

and so we obtain

$$e = \lim_{n \rightarrow \infty} \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\}.$$