This is the set of all real number pairs $(x, y)$ that satisfy (3). It is most remarkable and fortuitous that there are two points with integer coordinates lying in this region.

Although this example is probably too difficult for a precalculus class, it is quite useful for a numerical analysis or linear algebra course. It points out that a little round-off error can strongly affect the solution to a problem, when the solution procedure involves subtraction of almost equal quantities. It also provides a concrete example of an ill-conditioned linear system in which an input error of less than 0.3% yielded an output error over 10 times as large. Moreover, since the solutions are integers, students are forced to think about whether their solutions make sense in the context of the problem. Most of my students find only one of the solutions unless they are told that there are two possibilities.

This problem illustrates both the usefulness and the difficulty of interval analysis. Ramon E. Moore [2] discusses the solution of the linear system $Ax = b$ and distinguishes two cases: the first in which the coefficients of $A$ and $b$ are exactly representable by machine numbers, and the second in which the coefficients are only known to lie in certain intervals. He points out that the second case is "much more difficult" and that "the exact set of solutions... may be a complicated set." Moore cites a $2 \times 2$ example [1] for which the exact set of solutions is a nonconvex, eight-sided polygon. My example is somewhat easier to use in class, since the solution set is a convex polygon.

### References


1. Elrod Hansen, On the solution of linear algebraic equations with interval coefficients, *Linear Algebra and Applications* 2 (1969) 153–165.
2. Ramon E. Moore, *Methods and Applications of Interval Analysis*, SIAM, Philadelphia, 1979.
3. Edward Rozema, Why do we pivot in Gaussian elimination? *College Mathematics Journal* 19 (1988) 63–72.


## On the Distance from a Point to a Curve

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Let $C$ be a smooth curve (in the $x$-$y$ plane) parametrized by $\mathbf{r}(t)$ and let $q$ be a point not on $C$. Assume $q$ is the terminal point of the vector $\mathbf{q}$ in standard position, and let $f(t) = |\mathbf{r}(t) - \mathbf{q}|$, the distance from the terminal point $P$ of $\mathbf{r}(t)$ on $C$ to $q$. In this note, we shall determine the local extrema of $f$. The critical points of $f$ occur at the values of $t$ for which $(\mathbf{r}(t) - \mathbf{q}) \cdot \mathbf{r}'(t) = 0$. At such a point, if $q$ is on the convex side of $C$ (or $C$ is a line) then $f$ has a local minimum. Of greater interest is the case when $q$ is on the concave side of $C$; the result furnishes a nice application of the curvature and evolute of a curve. This case, depicted in Figure 1, is assumed in the following result.

**Theorem.** *Let $t_0$ be a critical point for the distance function $f$, and let $P_0$ be the corresponding point on $C$. Then $f$ has a local minimum (maximum) at $t_0$ if the distance from $q$ to $P_0$ is less (greater) than the radius of curvature at $P_0$.*

To establish the theorem, we start by recalling some formulas from vector calculus. A recommended reference is *Calculus with Analytic Geometry* by G. F.

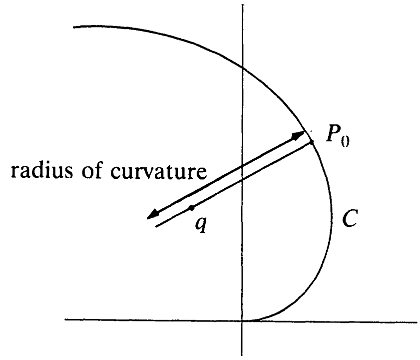


**Figure 1**

Simmons (New York, McGraw-Hill, 1985). Letting $P$ be the terminal point of $\mathbf{r}(t)$ on $C$, we have

$$\nu = \left|\frac{d\mathbf{r}}{dt}\right| = \frac{ds}{dt}$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}\frac{1}{\nu}$$

$$k\mathbf{N} = \frac{d\mathbf{T}}{ds}$$

where, at $P$, $s$ is the arc length along $C$ from any fixed point $P_0$, $\mathbf{T}$ is the unit tangent vector, $\mathbf{N}$ is the unit normal vector obtained by turning $\mathbf{T}$ counterclockwise through a right angle, and $k$ is the curvature. From this, we see that

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d(\mathbf{T}\nu)}{dt} = \frac{d\mathbf{T}}{dt}\nu + \frac{d\nu}{dt}\mathbf{T} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}\nu + \frac{d\nu}{dt}\mathbf{T}$$
$$= k\mathbf{N}\nu^2 + a\mathbf{T},$$

where $a = d\nu/dt$.

To classify the critical points of $f$, we consider the squared distance function $F = f^2 = |\mathbf{r} - \mathbf{q}|^2 = (\mathbf{r} - \mathbf{q}) \cdot (\mathbf{r} - \mathbf{q})$. Differentiating with respect to $t$, we get

$$F' = 2\mathbf{r}' \cdot (\mathbf{r} - \mathbf{q})$$

and

$$F'' = 2\mathbf{r}'' \cdot (\mathbf{r} - \mathbf{q}) + 2\mathbf{r}' \cdot \mathbf{r}' = -2(k\mathbf{N}\nu^2 + a\mathbf{T}) \cdot (\mathbf{q} - \mathbf{r}) + 2\nu^2.$$

Without loss of generality, we assume $k > 0$. Then, at a critical point $t_0$, we know $\mathbf{T} \cdot (\mathbf{q} - \mathbf{r}) = 0$ and $\mathbf{q} - \mathbf{r} = |\mathbf{q} - \mathbf{r}|\mathbf{N}$, the latter since $k\mathbf{N}$ always points to the concave side of $C$. Consequently, at $t_0$,

$$F'' = 2\nu^2(1 - k|\mathbf{q} - \mathbf{r}|)\begin{cases} > 0 & \text{if } |\mathbf{q} - \mathbf{r}| < 1/k \\ < 0 & \text{if } |\mathbf{q} - \mathbf{r}| > 1/k. \end{cases}$$

Recall, the radius of curvature $r$ at a point $P$ on a curve $C$ is given by $r = 1/|k|$

(assuming $k \neq 0$). Hence, a critical point $t_0$ provides a local minimum (maximum) if the distance from $q$ to $P_0$ is less (greater) than the radius of curvature at $P_0$.

A graphical approach becomes apparent when we introduce the *evolute* of $C$. Let $r$ be the radius of curvature of $C$ at a point $P$, and consider the normal line to $C$ at $P$, drawn to the concave side. Recall, the center of curvature of $C$ corresponding to $P$ is the point $Q$ on this normal whose distance from $P$ is $r$. The evolute $E$ is the locus of points generated by $Q$ as $P$ moves along $C$. The situation is depicted in Figure 2.

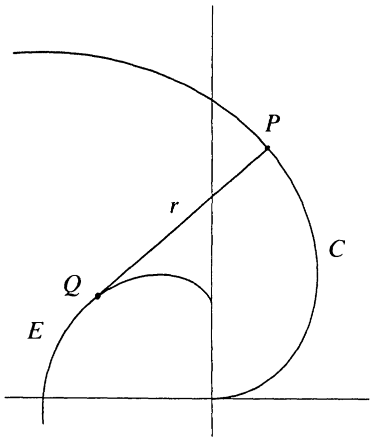


**Figure 2**

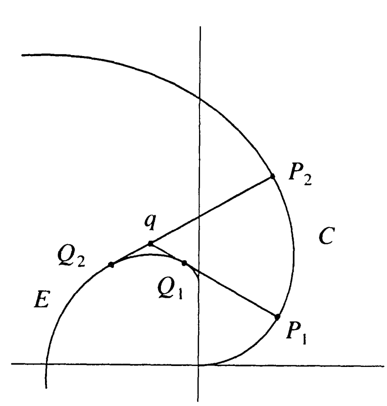


**Figure 3**

Armed with the evolute and the theorem above, we can visually classify the critical points of $f$. Let $P_1$ and $P_2$ be the points on $C$ corresponding to $t_1$ and $t_2$. In Figure 3, we see that $f$ has a local maximum at $t_1$ and a local minimum at $t_2$. This is because the distance from $q$ to $P_1$ is greater than the distance from $Q_1$ to $P_1$, the latter being the radius of curvature at $P_1$, and the distance from $q$ to $P_2$ is less than the distance from $Q_2$ to $P_2$, the latter being the radius of curvature at $P_2$.

A property of evolutes suggested by Figure 3 is that the normal line to $C$ at any point (drawn to the concave side) is tangent to the evolute. Simmons proves this on pp. 749–750. In principle, this provides a complete graphical determination (location and classification) of the critical points of $f$ by sketching all possible tangent lines to $E$ passing through $q$. Each such tangent line is normal to $C$, the point of intersection with $C$ determining a critical point of $f$.

Finally, for interested readers, the curve $C$ in the figures is the spiral $\mathbf{r}(t) = (2t \cos t, 2t \sin t)$. The evolute is easy to calculate and graph with a symbolic-graphing package.

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## Parametric Equations and Planar Curves

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One of the most satisfying fringe benefits of teaching mathematics is seeing unexpected and intriguing connections between various branches of mathematics while attempting to help others learn. In challenging students of multivariate