ARTICLES

The Lengthening Shadow: The Story of Related Rates

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Introduction

> A boy is walking away from a lamppost. How fast is his shadow moving?
>
> A ladder is resting against a wall. If the base is moved out from the wall, how fast is the top of the ladder moving down the wall?

Such "related rates problems" are old chestnuts of introductory calculus, used both to show the derivative as a rate of change and to illustrate implicit differentiation. Now that some "reform" texts [**4**, **14**] have broken the tradition of devoting a section to related rates, it is of interest to note that these problems originated in calculus reform movements of the 19th century.

Ritchie, related rates, and calculus reform

Related rates problems as we know them date back at least to 1836, when the Rev. William Ritchie (1790–1837), professor of Natural Philosophy at London University 1832–1837, and the predecessor of J. J. Sylvester in that position, published *Principles of the Differential and Integral Calculus*. His text [**21**, p. 47] included such problems as:

> If a halfpenny be placed on a hot shovel, so as to expand uniformly, at what rate is its *surface* increasing when the diameter is passing the limit of 1 inch and 1/10, the diameter being supposed to increase *uniformly* at the rate of .01 of an inch per second?

This related rates problem was no mere practical application; it was central to Ritchie's reform-minded pedagogical approach to calculus. He sought to simplify the presentation of calculus so that the subject would be more accessible to the ordinary, non-university student whose background might include only "the elements of

Geometry and the principles of Algebra as far as the end of quadratic equations." [**21**, p. v] Ritchie hoped to rectify what he saw as a deplorable state of affairs:

> The Fluxionary or Differential and Integral Calculus has within these few years become almost entirely a science of symbols and mere algebraic formulae, with scarcely any illustration or practical application. Clothed as it is in a transcendental dress, the ordinary student is afraid to approach it; and even many of those whose resources allow them to repair to the Universities do not appear to derive all the advantages which might be expected from the study of this interesting branch of mathematical science.

Ritchie's own background was not that of the typical mathematics professor. He had trained for the ministry, but after leaving the church, he attended scientific lectures in Paris, and "soon acquired great skill in devising and performing experiments in natural philosophy. He became known to Sir John Herschel, and through him [Ritchie] communicated [papers] to the Royal Society" [**24**, p. 1212]. This led to his appointment as the professor of natural philosophy at London University in 1832.

To make calculus accessible, Ritchie planned to follow the "same process of thought by which we arrive at actual discovery, namely, *by proceeding step by step from the simplest particular examples till the principle unfolds itself in all its generality.*" [**21**, p. vii; italics in original]

Drawing upon Newton, Ritchie takes the change in a magnitude over time as the fundamental explanatory concept from which he creates concrete, familiar examples illustrating the ideas of calculus. He begins with an intuitive introduction to limits through familiar ideas such as these: (i) the circle is the limit of inscribed regular polygons with increasing numbers of sides; (ii) $1/9$ is the limit of $1/10 + 1/100 + 1/1000 + \cdots$; (iii) $1/2x$ is the limit of $h/(2xh + h^2)$ as h approaches 0. Then—crucial to his pedagogy—he uses an expanding square to introduce both the idea of a function and the fact that a uniform increase in the independent variable may cause the dependent variable to increase at an increasing rate. Using Figure 1 to illustrate his approach, he writes:

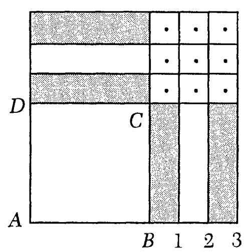

FIGURE 1
An expanding square

> Let AB be the side of a square, and let it increase uniformly by the increments $1, 2, 3$, so as to become $AB + 1$, $AB + 2$, $AB + 3$, etc., and let squares be described on the new sides, as in the annexed figure; then it is obvious that the square on the side $A1$ exceeds that on AB by the two shaded rectangles and the small white square in the corner. The square described on $A2$ has received an increase of two equal rectangles with *three* equal white squares in the corner. The square on $A3$ has received an increase of two equal rectangles and *five* equal small squares. Hence, when the side increases uniformly the area goes on at an increasing rate [**21**, p. 11].

Ritchie continues:

> The object of the differential calculus, is to determine the *ratio* between the rate of variation of the independent variable and that of the function into which it enters [**21**, p. 11].

A problem follows:

> If the side of a square increase uniformly, at what rate does the area increase when the side becomes x? [**21**, p. 11]

His solution is to let x become $x+h$, where h is the rate at which x is increasing. Then the area becomes $x^2+2xh+h^2$, where $2xh+h^2$ is the rate at which the area would increase if that rate were uniform. Then he obtains this proportion [**21**, p. 12]:

$$\frac{\text{rate of increase of the side}}{\text{rate of increase of area}} = \frac{h}{2xh+h^2}.$$

Letting h tend to zero yields $1/2x$ for the ratio.

He then turns to this problem:

> If the side of a square increase uniformly at the rate of three feet per second, at what rate is the area increasing when the side becomes 10 feet? [**21**, p. 12]

Using the previous result, he observes that since 1 is to $2x$ as 3 is to $6x$, the answer is 6×10. Then he expresses the result in Newton's notation: If $\dot{x}$ denotes the rate at which a variable x varies at an instant of time and if $u=x^2$, then $\dot{x}$ is to u as 1 is to $2x$, or $u=2x\dot{x}$.

In his first fifty pages, Ritchie develops rules for differentiation and integration. To illustrate the product rule, he writes:

> If one side of a rectangle vary at the rate of 1 inch per second, and the other at the rate of 2 inches, at what rate is the area increasing when the first side becomes 8 inches and the last 12? [**21**, p. 28]

His problem sets ask for derivatives, differentials, integrals, and the rate of change of one variable given the rate of change of another. Some related rates problems are abstract, but on pages 45–47 Ritchie sets the stage for the future development of related rates with nine problems, most of which concern rates of change of areas and volumes. One was the halfpenny problem; here are three more [**21**, p. 47–48]:

> 25. If the side of an equilateral triangle increase uniformly at the rate of 3 feet per second, at what rate is the area increasing when the side becomes 10 feet?...
>
> 30. A boy with a mathematical turn of mind observing an idle boy blowing small balloons with soapsuds, asked him the following pertinent question:—If the diameter of these balloons increase uniformly at the rate of $1/10$ of an inch per second, at what rate is the internal capacity increasing at the moment the diameter becomes 1 inch?...
>
> 34. A boy standing on the top of a tower, whose height is 60 feet, observed another boy running towards the foot of the tower at the rate of 6 miles an hour on the horizontal plane: at what rate is he approaching the first when he is 100 feet from the foot of the tower?

Since the next section of the book deals with such applications of the calculus as relative extrema, tangents, normals and subnormals, arc length and surface area, Ritchie clearly intended related rates problems to be fundamental, explanatory examples.

Augustus De Morgan (1806–1871) was briefly a professional colleague of Ritchie's at London University. De Morgan held the Chair of Mathematics at London University from 1828 to July of 1831, reassuming the position in October of 1836. Ritchie was appointed in January of 1832 and died in September of 1837. In *A Budget of Paradoxes*, published in 1872, De Morgan wrote [**9**, p. 296]:

> Dr. Ritchie was a very clear-headed man. He published, in 1818, a work on arithmetic, with rational explanations. This was too early for such an improvement, and nearly the whole of his excellent work was sold as waste paper. His elementary introduction to the Differential Calculus was drawn up while he was learning the subject late in life. Books of this sort are often very effective on points of difficulty.

De Morgan, too, was concerned with mathematics education. In *On the Study and Difficulties of Mathematics* [**6**], published in 1831, De Morgan used concrete examples to clarify mathematical rules used by teachers and students. In his short introduction to calculus, *Elementary Illustrations of the Differential and Integral Calculus* [**7**, p. 1–2], published in 1832, he tried to make calculus more accessible by introducing fewer new ideas simultaneously. De Morgan's book, however, does not represent the thoroughgoing reform that Ritchie's does. De Morgan touches on fluxions, but omits related rates problems. In 1836, shortly before Ritchie's death, De Morgan began the serial publication of *The Differential and Integral Calculus*, a major work of over 700 pages whose last chapter was published in 1842. He promised to make "the theory of limits... the sole foundation of the science, without any aid from the theory of series" and stated that he was not aware "that any work exists in which this has been avowedly attempted." [**8**, p. 1] De Morgan was more concerned with the logical foundations of calculus than with pedagogy; no related rates problems appear in the text.

Connell, related rates, and calculus reform

Another reform text appeared shortly after Ritchie's. James Connell, LLD (1804–1846), master of the mathematics department in the High School of Glasgow from 1834 to 1846, published a calculus textbook in 1844 promising "numerous examples and familiar illustrations designed for the use of schools and private students." [**5**, title page] Like Ritchie, Connell complained that the differential calculus was enveloped in needless mystery for all but a select few; he, too, proposed to reform the teaching of calculus by returning to its Newtonian roots [**5**, p. iv]. Connell wrote that he

> ...has fallen back upon the original view taken of this subject by its great founder, and, from the single definition of a rate, has been enabled to carry it out without the slightest assistance from Limiting ratio, Infinitesimals, or any other mode which, however good in itself, would, if introduced here, only tend to mislead and bewilder the student." [**5**, p. v]

To introduce an instantaneous rate, Connell asks the reader to consider two observers computing the speed of an accelerating locomotive as it passes a given point. One notes its position two minutes after it passes the point, the other after one minute; they get different answers for the speed. Instead of considering observations on shorter and shorter time intervals, Connell imagines the engineer cutting off the power at the given point. The locomotive then continues (as customary, neglecting friction) at a constant speed, which both observers could compute. This gives the locomotive's rate, or differential, at that point. Connell goes on to develop the calculus in terms of rates. For example, to prove the product rule for differentials, he considers the rectangular area generated as a particle moves so that its projections along the x- and y-axes move at the rates dx and dy respectively. As with Ritchie, the product rule is taught in terms of an expanding rectangle and rates of change.

Connell illustrates a number of the simpler concepts of the differential calculus using related rates problems. Some of his problems are similar to Ritchie's, but most are novel and original and many remain in our textbooks (punctuation in original):

> 5. A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. [**5**, p. 14]
>
> 6. One end of a ball of thread, is fastened to the top of a pole, 35 feet high; a person, carrying the ball, starts from the bottom, at the rate of 4 miles per hour, allowing the thread to unwind as he advances; at what rate is it unwinding, when the person is passing a point, 40 feet distant from the bottom of the pole; the height of the ball being 5 feet?...
>
> 12. A ladder 20 feet long reclines against a wall, the bottom of the ladder being 8 feet distant from the bottom of the wall; when in this position, a man begins to pull the lower extremity along the ground, at the rate of 2 feet per second; at what rate does the other extremity *begin* to descend along the face of the wall?...
>
> 13. A man whose height is 6 feet, walks from under a lamp post, at the rate of 3 miles per hour, at what rate is the extremity of his shadow travelling, supposing the height of the light to be 10 feet above the ground? [**5**, p. 20–24]

Connell died suddenly on March 26, 1847, leaving a wife and six children. The obituary in the *Glasgow Courier* observed that "he had the rare merit of communicating to his pupils a portion of that enthusiasm which distinguished himself. The science of numbers ... in Dr. Connell's hands ... became an attractive and proper study, and ... his great success as a teacher of children depended on his great attainments as a student of pure and mixed mathematics" [**26**]. It would be interesting to learn of any contact between Ritchie and Connell, but so far we have found none.

The rates reform movement in America

Related rates problems first appeared in America in an 1851 calculus text by Elias Loomis (1811–1889), professor of mathematics at Yale University. Loomis was also concerned to simplify calculus, writing that he hoped to present the material "in a more elementary manner than I have before seen it presented, except in a small volume by the late Professor Ritchie" [**17**, p. iv]. Indeed, the initial portion of

Loomis's text is essentially the same as Ritchie's. Loomis presents ten related rates problems, nine of which are Ritchie's; the one new problem asks for the rate of change of the volume of a cone whose base increases steadily while its height is held constant [**17**, p. 113]. Loomis's text remained in print from 1851 to 1872; a revision remained in print until 1902.

The next text to base the presentation of calculus on related rates was written by J. Minot Rice (1833–1901), professor of mathematics at the Naval Academy, and W. Woolsey Johnson (1841–1923), professor of mathematics at St. John's College in Annapolis. Where Loomis quietly approved the simplifications introduced by Ritchie, Rice and Johnson were much more enthusiastic reformers, drawing more from Connell than from Ritchie:

> Our plan is to return to the method of fluxions, and making use of the precise and easily comprehended definitions of Newton, to deduce the formulas of the Differential Calculus by a method which is not open to the objections which were largely instrumental in causing this view of the subject to be abandoned [**19**, p. 9].

In their 1877 text they derive basic differentiation techniques using rates. Letting dt be a finite quantity of time, dx/dt is the rate of x and "dx and dy are so defined that their ratio is equal to the ratio of the relative rates of x and y" [**20**, p. iv]. This approach has several advantages. First, it allows the authors to delay the definition of dy/dx as the limit of $\Delta y/\Delta x$ until Chapter XI, by which time the definition is more meaningful. Second, "the early introduction of elementary examples of a kinematical character ... which this mode of presenting the subject permits, will be found to serve an important purpose in illustrating the nature and use of the symbols employed" [**20**, p. iv].

These kinematical examples are related rates problems. Rice and Johnson use 26 related rates problems, scattered throughout the opening 57 pages of the text, to illustrate and explain differentiation. Rice and Johnson credit Connell in their preface and some of their problems resemble Connell's. Several other problems are similar to those of Loomis. However, Rice and Johnson also add to the collection of problems:

> A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it? [**20**, p. 28]
>
> Wine is poured into a conical glass 3 inches in height at a uniform rate, filling the glass in 8 seconds. At what rate is the surface rising at the end of 1 second? At what rate when the surface reaches the brim? [**20**, p. 37–38]

After Rice died in 1901, Johnson continued to publish the text until 1909. He was "an important member of the American mathematical scene ... [who] served as one of only five elected members of the Council of the American Mathematical Society for the 1892–1893 term" [**22**, p. 92–93]. The work of Rice and Johnson is likely to have inspired the several late 19th century calculus texts which were based on rates, focusing less on calculus as an analysis of tangent lines and areas and more on "how one quantity changes in response to changes in another." [**22**, p. 92]

James Morford Taylor (1843–1930) at Colgate, Catherinus Putnam Buckingham (1808–1888) at Kenyon, and Edward West Nichols (1858–1927) at the Virginia

Military Institute all wrote texts that remained in print from 1884 to 1902, 1875 to 1889, and 1900 to 1918, respectively. Buckingham, a graduate of West Point and president of Chicago Steel when his text was published was, perhaps, the most zealous of these reformist "rates" authors, believing that limits were problematic and could be avoided by taking rate itself as the primitive concept, much as he believed Newton did [**2**, p. 39].

Newton and precursors of the rates movement

Buckingham was correct that Newton conceived of magnitudes as being generated by motion, thereby linking calculus to kinematics. Newton wrote:

> I consider mathematical quantities in this place not as consisting of very small parts; but as described by a continued motion. Lines are described, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines .... These geneses really take place in the nature of things, and are daily seen in the motion of bodies .... Therefore considering that quantities which increase in equal times ... become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions ... and calling these velocities ... *fluxions*. [**3**, p. 413]

Since the 19th century reformers drew on Newton in revising the pedagogy of calculus, one wonders whether rates problems were part of an earlier tradition in England. The first calculus book to be published in English, *A Treatise of Fluxions or an Introduction to Mathematical Philosophy* [**13**] by Charles Hayes (1678–1760), published in 1704, treats fluxions as increments or decrements. Motion is absent and there are no related rates problems. But, in 1706, in the second book published in English, *An Institution of Fluxions* [**10**] by Humphrey Ditton (1675–1715), there are several problems which could be seen as precursors of related rates questions. While Ditton is interested in illustrating ideas of calculus using rates, he sticks to geometrical applications, not mechanical ones. He writes:

> A vast number of other Problems relating to the Motion of Lines and Points which are directly and most naturally solved by Fluxions might have been propos'd to the Reader. But this Field is so large, that 'twill be besides my purpose to do any more upon this Head than only just give some little Hints. [**10**, p. 172]

He gives one worked example. In FIGURE 2, b and c represent the new positions of points B and C respectively:

> If the Line AB, in any moment of Time be supposed to be divided into extream and mean Proportion, as ex. gr in the point C; then the Point A continues fixt, and the Points B and C moving in the direction AB, 'tis requir'd to find the Proportion of the Velocities of the points B and C; so that the flowing line Ab, may still be divided in extream and mean Proportion, e.g. in the Point c." [**10**, p. 171–172]

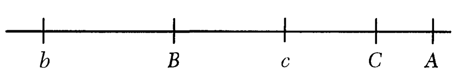

FIGURE 2
Points moving on a line

In his solution he lets $AB = y$, $AC = x$ and $BC = y - x$ and obtains $\frac{y}{y-x} = \frac{y-x}{x}$ giving $3yx = y^2 + x^2$. He differentiates, obtaining $3y\dot{x} + 3x\dot{y} = 2y\dot{y} + 2x\dot{x}$ which gives $\frac{\dot{x}}{\dot{y}} = \frac{2y-3x}{3y-2x}$. Ditton summarizes by saying, "the velocity of the Increment of the less Segment AC, must be the velocity of the Increment of the whole line AB as $\frac{2AB-3AC}{3AB-2AC}$" [**10**, p. 172]. His concern was to express ratios of rates of change of lengths in terms of ratios of lengths within the context of some geometric invariance. He was not seeking to use an equation involving rates of change as a centerpiece of pedagogy, but rather as a straightforward application of the calculus. His work did not lead to the development of such problems. Of the thirteen 18th century English authors surveyed, only William Emerson (1701–1782), writing in 1743, included a related rates problem, but not in a significant way [**11**, p. 108].

In the early 19th century we find scattered related rates problems. There is a sliding ladder problem in a Cambridge collection: "The hypotenuse of a right-angled triangle being constant, find the corresponding variations of the sides." [**25**, p. 678] John Hind's text included one problem: "Corresponding to the extremities of the *latus rectum* of a common parabola, it is required to find the ratio of the rates of increase of the abscissa and ordinate" [**14**, p. 148]. Neither of these problems plays the important pedagogical role that we find in the works of Ritchie or Connell.

The twilight of related rates

Why did so few books illustrate calculus concepts using related rates problems? One reason is that from the beginning of the 18th century to the middle of the 19th century, the foundations of calculus were hotly debated, and Newton's fluxions did not compete very successfully against infinitesimals, limits, and infinite series. Among those who chose to base calculus on fluxions, many still felt uneasy about including kinematical considerations in mathematics. In *A Comparative View of the Principles of the Fluxional and Differential Methods*, Prof. D.M. Peacock wrote that one of the leading objections to the fluxional approach was that "it introduces Mechanical considerations of *Motion*, *Velocity*, and *Time*, foreign to the genius of pure Analytics" [**18**, p. 6]. Such concepts were considered by some to be "inconsistent with the rigour of mathematical reasoning, and wholly foreign to science." [**23**, p. 7]

In England, moreover, resistance to Newton's approach to calculus as well as to the French approach as expressed in Lacroix's textbook [**16**] rested in part on the belief that the purpose of mathematics was to train the mind. That meant doing calculus within a Euclidean framework with a clear focus on the properties of geometrical figures [**1**, p. v–xx].

By the end of the 19th century, most authors were developing calculus on the basis of limits. In the works of Simon Newcomb (1835–1909) and Edward Bowser (1845–1910), for example, related rates problems illustrate the derivative as a rate of change, but the problems are not central. In 1904, William Granville (1863–1943) published his *Elements of the Differential and Integral Calculus*, which remained in print until 1957. This text, which introduced concepts intuitively before establishing analytical arguments, became the standard by which other texts were measured. In the 1941 edition, Granville laid out a method for solving related rates problems, but these problems had now become an end in themselves rather than an exciting and pedagogically important method by which to introduce calculus.

Conclusion

An informal survey of ours suggests that for many teachers these problems have lost their significance. One often hears teachers say that related rates problems are contrived and too difficult for contemporary students. It is thus ironic that such problems entered calculus through reformers who believed, much as modern reformers do, that in order for calculus to be accessible, concrete, apt illustrations of the derivative are necessary. Twilight for our 19th century reformers would have suggested a lengthening, accelerating shadow, not the end of an era. They might well have written:

Related rates, a pump, not a filter; a sail, not an anchor.

NOTE. See `http://www.maa.org/pubs/mm-supplements/index.html` for a more extensive bibliography.

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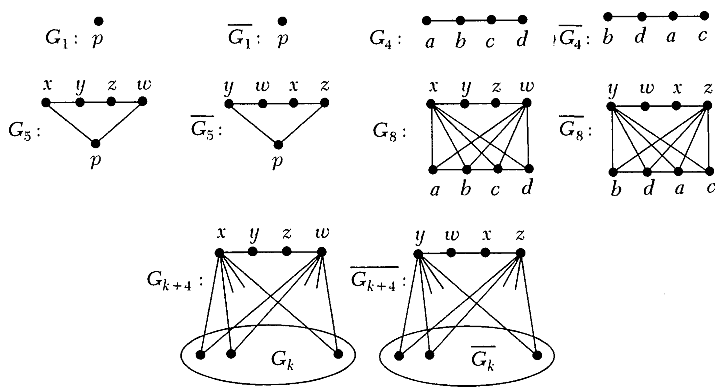
Proof Without Words: Self-Complementary Graphs

A graph is *simple* if it contains no loops or multiple edges. A simple graph $G = (V, E)$ is *self-complementary* if G is isomorphic to its *complement* $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{\{v, w\} : v, w \in V,\ v \neq w, \text{ and } \{v, w\} \notin E\}$. It is a standard exercise to show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. A converse also holds, as we shall now show.

THEOREM. *If n is a positive integer and either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then there exists a self-complementary simple graph G_n with n vertices.*

Proof.



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