NOTES

Indeterminate Forms Revisited

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1. Introduction You must all have seen at least one calculus textbook. It may surprise some of you that three centuries ago no such book existed: the very first book that was in any sense a calculus text was published, anonymously, in 1696, under the rather forbidding title *Analysis of the Infinitely Small* [**5**]. It was well known in European mathematical circles that the author was a French marquis, Guillaume de L'Hôpital. (I give him the modern French spelling, which at least keeps students from pronouncing the silent s in L'Hospital.) The book was hardly easy reading. It began with propositions like this: "One can substitute, one for the other, two quantities which differ only by an infinitely small quantity; or (what amounts to the same thing) a quantity that is increased or decreased only by another quantity infinitely less than it, can be considered as remaining the same." This sort of presentation gave calculus a reputation, which has survived to modern times, of being unintelligible.

Sylvester, writing in about 1880 [**10**, vol. 2, pp. 716-17], said that when he was young (around 1830) "a boy of sixteen or seventeen who knew his infinitesimal calculus would have been almost pointed out in the streets as a prodigy like Dante, who had seen hell." (Here and now, we would very likely find students of the same age feeling much the same; but Sylvester, in 1870, was teaching students who dealt casually with topics that we would now describe as advanced calculus.) When I was in high school (somewhat later), calculus was thought of, by otherwise well-educated people, as being as deep and mysterious as (say) general relativity is thought of today. My parents knew somebody who was reputed to know calculus, but they had no idea what that was (and they were college teachers—of English). Nowadays there are perhaps too many calculus books, but some of the answers that students give to examination questions make me wonder whether the subject has even now become sufficiently intelligible.

In his own time, and for long afterwards, L'Hôpital had an impressive reputation. Today he is remembered only for "L'Hôpital's rule," which evaluates limits like

$$\lim_{x \to 1} \frac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{3/4}}$$

(L'Hôpital's own example) by replacing the numerator and denominator by their derivatives and hoping for the best.

L'Hôpital's rule seems to have fallen somewhat out of favor; I have heard it claimed that all it is useful for is as an exercise in differentiation.

It has been known for some time that many of L'Hôpital's results, including the rule, were purchased (quite literally) from John (= Jean = Johann) Bernoulli. Immediately after L'Hôpital's death in 1704, Bernoulli published an article claiming that he had communicated the rule for $0/0$ to L'Hôpital, along with other material, before L'Hôpital had published it. This claim was disbelieved for some two hundred years; sceptics wondered why Bernoulli had not advanced his claim earlier. The reason for the delay eventually became clear when Bernoulli's correspondence with L'Hôpital came to light in the early 1900s. Bernoulli gave the rule to L'Hôpital only after L'Hôpital had promised to pay for it, had repeatedly asked for it, and had finally come across with the first installment. We now also know that there are records of Bernoulli's having lectured on the rule before L'Hôpital's book was published.

In the preface to his book, L'Hôpital says, "I acknowledge my debt to the insights of MM Bernoulli, above all to those of the younger [John], now Professor at Groningen. I have unceremoniously made use of their discoveries and of those of M Leibnis [sic]. Consequently I invite them to claim whatever they wish, and will be satisfied with whatever they may leave me." Considering what we now know, this seems somewhat disingenuous, especially since L'Hôpital was clearly unable to discover for himself how to prove the rule of which, as Plancherel once said of his own theorem, he had "the honor of bearing the name."

You can find the whole story in the 1955 volume of Bernoulli's correspondence **[7]**, or in Truesdell's review of the volume **[11]**.

I used to wonder, from time to time, what kind of proof L'Hôpital had used, but never when I was where I could find a copy of his book. Recently I happened to mention this question to Professor Alexanderson—who promptly produced his own copy. Professor Underwood Dudley, who is more resourceful than I am, also found a copy, and has translated it into modern terminology **[4]**, but retaining its geometric character (L'Hôpital thought of functions as curves). L'Hôpital actually considered only $\lim_{x\to a} f(x)/g(x)$, where a is finite, $f(a)=g(a)=0$, and both $f'(a)$ and $g'(a)$ exist, are finite, and not zero. In analytical language, what L'Hôpital did amounts to writing

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{f'(a)+\varepsilon(x)}{g'(a)+\delta(x)}(\varepsilon\to 0,\delta\to 0)=\frac{f'(a)}{g'(a)}.$$

It is not trivial to extend such a proof to the cases when $f'(a)$ and $g'(a)$ do not exist (but have limits as $x \to a$), or are both zero, or $f(a)=g(a)=\infty$, or $a=\infty$. I do not know when or by whom these refinements were added, but the complete theory was in place by 1880 **[8, 9]**.

2. A common modern proof If you saw a proof of L'Hôpital's rule in a modern calculus class, the probability is about 90% that it is Cauchy's proof. This proof appeals to mathematicians because it is elegant, but often fails to appeal to students because it is subtle. It depends on knowing Cauchy's refinement of the mean-value theorem, namely that (with appropriate hypotheses)

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)},\quad c \text{ between } a \text{ and } b. \tag{1}$$

Given this, L'Hôpital's rule becomes obvious.

In spite of its elegance, Cauchy's proof seems to me to be inappropriate for an elementary class. Any proof that begins with a lemma like Cauchy's mean value theorem, that says "Let us consider...," repels most students. Students are also uncomfortable with the nebulous point c. They want to know where it is, and feel that the instructor is deliberately keeping them in the dark. Of course, the exact location of c is completely irrelevant (although numerous papers have been written about it).

3. A caution Cauchy's proof tacitly assumes that there is a (one-sided) neighborhood of the point a in which $g'(x) \neq 0$. Strictly speaking, if there is no such neighborhood, the limit in (1) is not defined, and we would have no business talking about it. However, if f' and g' are given by explicit formulas, they may happen to share a common factor that is zero at a, and the temptation to cancel this factor is irresistible. One can obtain a spurious result in this way [**8**, **9**; **3**, p. 124, ex. 24; **1**].

Let me give you a specific example, just to emphasize that there is a reason for the requirement that $g'(x) \neq 0$. Let $f(x) = 2x + \sin 2x$, $g(x) = x \sin x + \cos x; a = +\infty$. Then $f'(x) = 4\cos^2 x$, $g'(x) = x \cos x$, and $f'(x)/g'(x) \to 0$, whereas $f(x)/g(x)$ does not approach a limit. The trouble comes from cancelling a factor that changes sign in every neighborhood of the point a; it would have been legitimate to cancel a quadratic factor.

Some writers think that the difficulty arises only in artificial cases that would never occur in practice. But then, what happens to our claim to be giving correct proofs?

You might not guess from Cauchy's proof that there is a discrete analog of L'Hôpital's rule; see, for example, [**6**]. This was known to Stolz in the 1890's, and has often been rediscovered.

4. A more satisfactory proof I want now to show you a proof of L'Hôpital's rule that avoids the difficulties of Cauchy's and establishes a good deal more. It may seem more complicated, but not if you include a proof of Cauchy's mean value theorem as part of Cauchy's proof. This proof is also quite old; Stolz knew it, but preferred Cauchy's proof, perhaps because of Cauchy's reputation. It has been published several times by people (including me) who failed to search the literature.

Let us suppose that $f(x)$ and $g(x)$ approach 0 as $x \to a$ from the left, where a might be $+\infty$; it does no harm to define (if necessary) $f(a) = g(a) = 0$. We may suppose that $g'(x) > 0$ (otherwise consider $-g(x)$). Now let $f'(x)/g'(x) \to L$, where $0 < L < \infty$. Then, given $\varepsilon > 0$, we have, if x is sufficiently near a, and a is finite,

$$L - \varepsilon \leqslant \frac{f'(x)}{g'(x)} \leqslant L + \varepsilon,$$

$$(L - \varepsilon) g'(x) \leqslant f'(x) \leqslant (L + \varepsilon) g'(x) \quad (\text{since } g'(x) > 0).$$

Integrate on (x, a) to get

$$-(L - \varepsilon) g(x) \leqslant -f(x) \leqslant -(L + \varepsilon) g(x)$$

(notice that since g increases to 0, we have $g(x) < 0$). Since $-g$ and $-f$ are positive near a,

$$L - \varepsilon \leqslant \frac{f(x)}{g(x)} \leqslant L + \varepsilon,$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Only formal changes are needed if $a = +\infty$ or if $L = 0$ or ∞.

For the ∞/∞ case, we get, in the same way, with $\delta > 0$,

$$L-\varepsilon<\frac{f(a-\delta)-f(x)}{g(a-\delta)-g(x)}<L+\varepsilon,$$

$$L-\varepsilon<\frac{\dfrac{f(a-\delta)}{f(x)}-1}{\dfrac{g(a-\delta)}{g(x)}-1}\cdot\frac{f(x)}{g(x)}<L+\varepsilon.$$

Letting $x \to a$, we obtain

$$L-\varepsilon \leqslant \liminf_{x\to a}\frac{f(x)}{g(x)} \leqslant \limsup_{x\to a}\frac{f(x)}{g(x)} \leqslant L+\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $f(x)/g(x) \to L$.

If it happens that $f'(a) = g'(a) = 0$, one repeats the procedure with f'/g', and so on. If $f^{(n)}(a) = g^{(n)}(a) = 0$ for every n (which can happen with f and g not identically zero), the procedure fails. Otherwise, the limit can be handled more simply in a single step, as we shall see below.

5. Generalizations If f and g are defined only on the positive integers, we can reason in a similar way with differences instead of derivatives to conclude that if the differences of g are positive, and $f(n)$ and $g(n)$ approach zero as $n \to \infty$, then if

$$\frac{f(n)-f(n-1)}{g(n)-g(n-1)} \to L \qquad \text{as } n \to \infty,$$

it follows that $f(n)/g(n) \to L$. This is sometimes called Cesàro's rule. For more detail, and illustrations, see **[6]**. A possibly more familiar version is as follows:

If $a_n \to 0$ and $b_n \to 0$, and $a_n/b_n \to L$, then

$$\frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} \to L.$$

The key point in the proof of L'Hôpital's rule is the principle that the integral of a nonnegative function ($\not\equiv 0$) is positive. More precisely, if $f(x) \geqslant 0$ on $p \leqslant x \leqslant q$ then

$$\int_p^q f(t)\,dt > 0 \qquad \text{if } p < x < q \text{ and } f(t) \not\equiv 0.$$

Repeated integration with the same lower limit has the same property, as we see by rewriting the n-fold iterated integral as a single integral:

$$\frac{1}{(n-1)}!\int_p^x (t-p)^{n-1} f(t)\,dt.$$

This suggests the appropriate treatment of the case of L'Hôpital's rule when $f'(a) = g'(a) = 0$, or more generally when $f^{(k)}(a) = g^{(k)}(a) = 0, k = 1, 2, \ldots, n-1$, but not both of $f^{(n)}(a)$ and $g^{(n)}(a)$ are 0. The positivity of iterated integration then yields the conclusion of L'Hôpital's rule in a single step.

An operator that carries positive functions to positive functions is conventionally called a positive operator. If F is an invertible operator whose inverse is positive, we can conclude that if $F[f(x)]/F[g(x)] \to L$ and $F[g] > 0$, then $f(x)/g(x) \to L$.

As an example of the use of operators, consider $D + P(x)I$, where $D = d/dx$ and I is the identity operator. This is the operator that occurs in the theory of the linear first-order differential equation $y' + P(x)y = Q(x)$. The solution of this differential equation, with $y(a) = 0$, is

$$y = \exp\left\{\int_a^x P(t)\,dt\right\} \int_a^x Q(t)\exp\left\{\int_a^t P(u)\,du\right\} dt. \tag{2}$$

In other words, (2) provides the inverse Λ of $D + P(x)I$.

The explicit formula shows that if $Q(x) \geqslant 0$ we have $\Lambda[Q] > 0$, so that Λ is a positive operator. Hence we may conclude that if

$$\frac{[D+PI]f}{[D+PI]g} \to L$$

and $[D + PI]g > 0$ then

$$(L-\varepsilon)[D+PI]g < [D+PI]f < (L+\varepsilon)[D+PI]g.$$

A positive linear operator evidently preserves inequalities. Consequently, if we apply Λ to both sides, we obtain

$$(L-\varepsilon)g(x) < f(x) < (L+\varepsilon)g(x)$$

and hence

$$f(x)/g(x) \to L.$$

Thus $D + P(x)I$ can play the same role as D in L'Hôpital's rule. It is at least possible that $D + P(x)I$ might be simpler than D.

Since some forms of fractional integrals and derivatives are defined by positive operators, one could also formulate a fractional L'Hôpital's rule.

This article is the text of an invited address to a joint session of the American Mathematical Society and the Mathematical Association of America, January 14, 1989.

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