edges, that is, $|E(Q_n)| = n2^{n-1}$. The distance between vertices x and y is given by $\sum_{i=1}^{n} |x_i - y_i|$, that is, the number of place disagreements in their binary vectors.

Calling the vertex $(0, 0, \ldots, 0)$ the *origin*, define the ith *distance set* D_i, as the set of vertices whose distance from the origin is i. Then for each $i = 0, 1, 2, \ldots, n$, we have $D_i = \{(x_1, x_2, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = i\}$, that is, D_i consists of those vertices with exactly i 1s in their binary n-vectors. Moreover, we have $|D_i| = \binom{n}{i}$. The fact that the D_is partition $V(Q_n)$ demonstrates Equation (2) rather nicely.

Now observe that the induced subgraph on any D_i contains no edges, since all of the binary vectors of the vertices in D_i contain the same number of 1s. (If two vertices are adjacent, the number of 1s in their binary vectors must differ by exactly one.) Furthermore, if $|i - j| \geq 2$, then if $x \in D_i$ and $y \in D_j$, it follows that x and y are nonadjacent, that is, $xy \notin E(Q_n)$. Then all edges are of the form uv, where $u \in D_i$ and $v \in D_{i+1}$, for $i = 0, 1, 2, \ldots, n-1$. Since each vertex in D_{i+1} has $i+1$ 1s in its binary vector, it is adjacent to exactly $i+1$ vertices in D_i. (These vertices are obtained by replacing one 1 by 0 in the binary vector of the chosen vertex in D_{i+1}.) This implies that the number of edges with endpoints in both D_i and D_{i+1} is $(i+1)|D_{i+1}| = (i+1)\binom{n}{i+1}$. It follows that the total number of edges in Q_n is given by $\sum_{i=0}^{n-1} (i+1)\binom{n}{i+1}$. Finally, since $|E(Q_n)| = n2^{n-1}$, we are done with the proof of (4).

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A Derivation of Taylor's Formula with Integral Remainder

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Taylor's formula with integral remainder is usually derived using integration by parts [**4, 5**], or sometimes by differentiating with respect to a parameter [**1, 2**]. According to M. Spivak [**7**, p. 390], integration by parts is applied in a "rather tricky way" to derive Taylor's formula, using a substitution that "one might discover after sufficiently many similar but futile manipulations". In this MAGAZINE, Lampret [**3**] derived both Taylor's formula and the Euler-Maclaurin summation formula using a rather heroic application of integration by parts.

We derive the remainder formula in a way that avoids tricks and heroics. The key step is changing the order of integration in multiple integrals, a topic that many students in an analysis class will benefit from reviewing. This derivation has almost certainly been found many times before [**6**], however, most people seem to be unaware of it.

The Taylor formula Suppose that a function $f(x)$ and all its derivatives up to $n+1$ are continuous on the real line. Then Taylor's formula for $f(x)$ about 0 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R(x), \tag{1}$$

where the remainder, $R(x)$, is given by

$$R(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du.$$

Our derivation is based on the following simple idea: Try to reconstruct f by integrating $f^{(n+1)}$, $n+1$ times. This approach is suggested by the case $n=0$, when (1) is merely the fundamental theorem of calculus. For notational simplicity, we prove (1) for only $n=2$; however, the general case is similar. Thus, consider

$$\widetilde{R}(x) := \int_0^x \int_0^w \int_0^v f^{(3)}(u)\,du\,dv\,dw. \tag{2}$$

Now let's evaluate this integral in two ways. The first way is by direct integration using the fundamental theorem of calculus three times:

$$\widetilde{R}(x) = f(x) - f(0) - xf'(0) - \frac{x^2}{2!}f''(0). \tag{3}$$

The second way to integrate (2) is by interchanging the order of integration:

$$\int_0^w \int_0^v f^{(3)}(u)\,du\,dv = \int_0^w \int_u^w f^{(3)}(u)\,dv\,du = \int_0^w (w-u)f^{(3)}(u)\,du.$$

Interchanging the order of integration again gives

$$\begin{aligned}\int_0^x \left\{\int_0^w \int_0^v f^{(3)}(u)\,du\,dv\right\} dw &= \int_0^x \left\{\int_0^w (w-u)f^{(3)}(u)\,du\right\} dw \\ &= \int_0^x \int_u^x (w-u)f^{(3)}(u)\,dw\,du \\ &= \frac{1}{2}\int_0^x (x-u)^2 f^{(3)}(u)\,du. \end{aligned} \tag{4}$$

Equating (3) and (4) yields the Taylor formula (1) for $n=2$.

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A Theorem Involving the Denominators of Bernoulli Numbers

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The Swiss mathematician, Jakob Bernoulli (1654–1705), successfully sought a general method for summing the first n kth powers for arbitrary positive integers n and k. Let us define

$$S_k(n) = \sum_{j=1}^{n} j^k = 1^k + 2^k + \cdots + n^k.$$

Define the average of the first n kth powers by

$$\mu_k(n) = \frac{S_k(n)}{n}.$$

We pose and answer the following natural question: For which values of n and k is $\mu_k(n)$ an integer? Our answer, although it does involve the denominators of Bernoulli numbers, which undergraduates may not have seen, relies primarily upon elementary divisibility arguments.

Background In his *Ars Conjectandi*, published posthumously in 1713 and dedicated primarily to the theory of probability, Bernoulli presented a recursive solution for $S_k(n)$. It states that for $k \geq 1$,

$$(n+1)^{k+1} = (n+1) + \sum_{j=1}^{k} \binom{k+1}{j} S_j(n),$$

where the binomial coefficients are defined as usual:

$$\binom{k+1}{j} = \frac{(k+1)!}{j!\,(k+1-j)!}.$$