NOTES

A Diophantine Equation from Calculus

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1. Introduction

In constructing exercises for homework and tests one often tries to choose constants in a problem so that the answer has a simple numerical form. For one frequently used calculus problem this effort leads to a Diophantine equation, equation (1) below. General solutions of this type of equation are well-known [**2**], [**4**], [**6**], [**7**], and especially see [**3**, pages 405–406]. In this investigation we will be interested in certain kinds of solutions. Also we will make use of a symmetry of the equation and the solutions which seems not to have been noticed previously. To this end we will develop the solution in a way that facilitates the use of the symmetry. Furthermore we include a table of certain solutions suitable for new variations of the original calculus problem.

2. The Problem

The problem is the familiar one from calculus: Construct an open box from a rectangular piece of tin of dimensions a and b by cutting equal squares of edge x from the corners and bending up the resulting tabs. Find x which maximizes the volume of the box.

The dimensions a and b in use are (a, a), any number; integer multiples of $(a, b) = (5,8)$; integer multiples of $(a, b) = (8,15)$; and integer multiples of $(a, b) = (16,21)$. No texts [**1**], [**5**], [**8**], [**9**], [**10**], [**11**], [**12**], [**13**] that we have found use any other dimensions that will lead to a rational value for x, yet there are many such pairs.

The volume V of the resulting box has formula $V = x(a-2x)(b-2x)$, and has maximum value when $x = [(a+b) - \sqrt{a^2+b^2-ab}\,]/6$. Thus x is rational when a, b, and c are integers with

$$a^2 + b^2 - ab = c^2. \tag{1}$$

In particular we examine the solutions in which a, b, and c are positive, relatively prime integers with $a < b$. We call these the *primitive* solutions.

3. Square Solutions

When $a = b$, (1) becomes $a^2 = c^2$ and every pair of integers (a, a) provides the rational solution for x, $x = a/6$. This is not very interesting.

4. Interesting Solutions

Now we consider solutions with $a < b$. There is a symmetry in the set of solutions, namely if (a, b, c) is a solution then so is $(b - a, b, c)$. This is easily verified by substitution in (1). It is also easy to verify that when (a, b, c) is primitive then $(b - a, b, c)$ is primitive, and different from the given solution.

Equation (1) is transformed by

$$\begin{aligned} a &= n - m \\ b &= n + m \end{aligned} \tag{2}$$

to obtain

$$n^2 + 3m^2 = c^2 \tag{3}$$

(Such a transformation follows standard procedure for simplifying quadratic forms.)

The inverse transformation to (2) is

$$\begin{aligned} m &= (b - a)/2 \\ n &= (b + a)/2, \end{aligned} \tag{4}$$

from which it follows that $0 < m < n$. The primitive solutions of (1) are related in a simple way to the primitive solutions of (3).

In discussing this relationship it is convenient to distinguish between solutions (1) (or (3)) in which a and b (m and n) are both odd, and those in which a and b (m and n) have opposite parity. The first we call *pure*, the second *mixed*.

CASE 1: The mixed primitive solutions of (3) correspond to the pure primitive solutions of (1).

Let (m, n) be a mixed primitive solution of (3) corresponding to the solution $a = n - m$, $b = n + m$. Clearly $0 < a < b$ and a and b are both odd. A common divisor of a and b must be odd, hence it will be a common divisor of $m = (b - a)/2$ and $n = (b + a)/2$. The only such divisor is 1, hence (a, b, c) is a pure primitive solution of (1).

When (a, b) is a pure primitive solution of (1) a similar argument will show that $m = (b - a)/2$, $n = (b + a)/2$ is a mixed primitive solution of (3), establishing the correspondence in this case.

CASE 2: The pure primitive solutions of (3) correspond to two times the mixed primitive solutions of (1).

Let (m, n, c^*) be a pure primitive solution of (3). Then $(n - m, n + m, c^*)$ is an integral solution of (1) in which $0 < n - m < n + m$. However, it is clear that each number is even since (n, m) is pure. Let $a = (n - m)/2$, $b = (n + m)/2$, and $c = c^*/2$. This is a positive integral solution of (1) which is mixed, otherwise, as in case (1), it would correspond by (4) to the mixed integral solution of (3), $(m/2, n/2, c^*/2)$. This would contradict the primitivity of (m, n, c^*).

This solution of (1) is, in fact, primitive, for any divisor of a, b, c will also divide $m = b - a$, $n = b + a$, and $c^* = 2c$, a sequence whose only common divisor is 1. Thus the pure primitive solution (m, u, c^*) of (3) corresponds to the mixed primitive solution $(a = (n - m)/2, b = (n + m)/2, c = c^*/2)$ of (1).

Starting with a mixed primitive solution of (1) one sees, through a similar argument, that such a solution corresponds through (4) to one-half a pure primitive solution of (3).

We will use the method given in [7] and the symmetry mentioned above for investigating the primitive solutions of (3), and hence of (1). First factor $3m^2 = c^2 - n^2$, obtaining $3m^2 = (c-n)(c+n)$. Let $d = \gcd(c-n, c+n)$. Then $3m^2 = rsd^2$ where $c - n = rd$, $c + n = sd$, $r < s$, and r and s are relatively prime. Furthermore, one of r and s is a square and the other is three times a square.

The factor d must be either 1 or 2. To see this suppose that $d \neq 1$. Now $d^2 | 3m^2$, so if $3|d$ it follows that $3|m$. But $d|[(c+n)-(c-n)] = 2n$, and if $3|d$, then $3|2n$ whence $3|n$. This is impossible since m and n are relatively prime. Therefore $d^2|m^2$ and $d|m$. From above we also have $d|2n$. Since $\gcd(m,n) = 1$ we conclude that $d = 2$.

Next we readily see that from $3|3m^2 = rsd^2$ and $d = 1$ or 2, one can conclude that $3|r$ and $3 \nmid s$ or $3 \nmid r$ and $3|s$. First consider the case in which $3|r$.

In this kind of solution

$$m^2 = (r/3)sd^2, \tag{5}$$

with $r/3$ and s relatively prime and both squares of integers, say $r/3 = u^2$ and $s = v^2$ where $\gcd(3u, v) = 1$. Substituting in (5) results in $m^2 = u^2v^2d^2$ whence

$$m = uvd, \quad 3u < v, \quad \gcd(3u, v) = 1. \tag{6}$$

Recalling that $2n = (c+n) - (c-n) = v^2d - 3u^2d$, we have

$$n = (v^2 - 3u^2)d/2. \tag{7}$$

When $d = 1$, v and u must both be odd. But when $d = 2$ then v and u are of opposite parity, for otherwise m and n would both be even.

Equations (6) and (7) express m and n in terms of u, v, and d. Substitution in (2) yields

$$\begin{aligned} a &= (v^2 - 2uv - 3v^2)d/2 \\ b &= (v^2 + 2uv - 3v^2)d/2. \end{aligned} \tag{8}$$

The conditions on u and v are $0 < 3u < v$ and $\gcd(3u, v) = 1$. When $d = 1$, u and v are both odd giving a pure primitive solution (m, n) for (3). Then (8) is twice a mixed primitive solution of (1). When $d = 2$, u and v have opposite parity, giving a mixed primitive solution of (3). The solution of (1) given by (8) is then a pure primitive solution.

All such choices of u, v, and d yield half of the primitive solutions of (3) and thus of (1). The other half, in which $3 \nmid r$ and $3|s$, can be found by a similar analysis. However the symmetry of (1) observed earlier interchanges these two kinds of primitive solutions. For $(a, b, c) \to (b - a, b, c)$ induces $(m, n, c) \to ((n-m)/2, (n+3m)/2, c)$ for solutions of (3). Then

$$\begin{aligned} c - (n+3m)/2 &= [c + (c-n) - 3m]/2 \\ c + (n+3m)/2 &= [3c - (c-n) + 3m]/2. \end{aligned} \tag{9}$$

If $3|c - n$ (a solution of the first kind), primitivity of (m, n, c) implies that $3 \nmid c$, and from (9) it follows that $3 \nmid c - (n+3m)/2$, and 3 does divide $c + (n+3m)/2$. The new solution is therefore of the second kind.

TABLE 1 lists some solutions of the first kind, obtained by taking values of u, v, and d. TABLE 2 lists transformed solutions. Each table provides new integer pairs a and b for dimensions for the rectangular box problem.

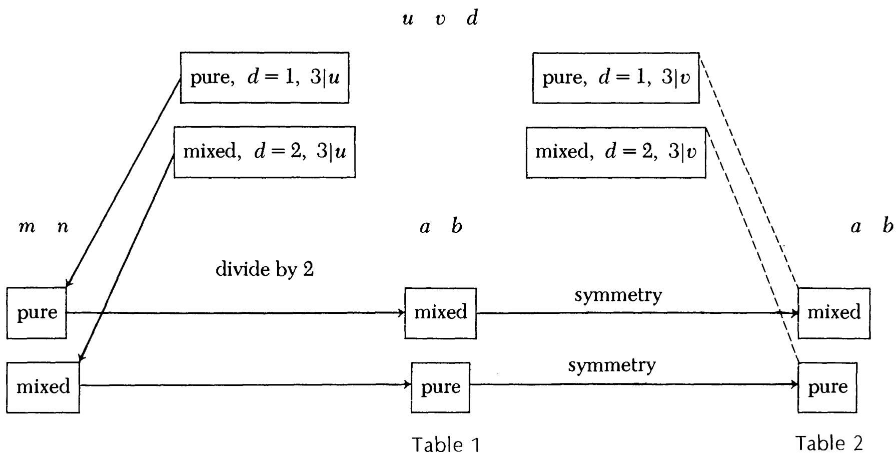
TABLE 1

u	v	d	m	n	a	b	c
1	5	1	5	11	3	8	7
1	7	1	7	23	8	15	13
3	11	1	33	47	7	40	37
5	17	1	85	107	11	96	91
7	23	1	191	161	15	176	169
1	4	2	8	13	5	21	19
2	7	2	28	37	9	65	61
3	10	2	60	73	13	133	127
4	13	2	104	121	17	225	217

TABLE 2

a	b	c
5	8	7
7	15	13
33	40	37
85	96	91
161	176	169
16	21	19
56	65	61
120	133	127
208	225	217

A diagram relating the values of u, v, d with (m, n) and in turn with (a, b) is given below.



5. A Final Comment

The symmetry observed in the solutions of (1) is a special case of a symmetry for equations of the form

$$a^2 + b^2 - qab = c^2. \tag{10}$$

The symmetry is $(a, b, c) \to (qb - a, b, c)$, a symmetry of order 2.

By the substitution

$$\begin{aligned} a &= (q^2 - 2)m + qn \\ b &= qm + n \end{aligned} \tag{11}$$

equation (9) changes to

$$(4 - q^2)m^2 + n^2 = c^2 \tag{12}$$

and the corresponding symmetry for solutions of (12) is

$$(m, n, c) \to \left(\left[(q^2 - 2)m + qn\right]/2, \left[(q^3 - 4q)m + (q^2 - 2)n\right]/2, c\right).$$

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How Small Is a Unit Ball?

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The volume of the cube of edge d in $\mathbb{R}^n$ is d^n so that, as the dimension n increases, this volume increases, stays constant, or decreases to zero according as $d>1$, $d=1$, or $d<1$. The situation for the ball of radius r in $\mathbb{R}^n$ is quite different.

For $n=0,1,2,\ldots,$ and $r>0$, let $V_n(r)$ denote the n-dimensional volume of the n-dimensional ball of radius r in $\mathbb{R}^n$. Then $V_0(r)=1$, $V_1(r)=2r$, $V_2(r)=\pi r^2$, $V_3(r)=(4/3)\pi r^3$, and in general

$$V_n(r)=\frac{r^n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

The derivation of this formula for $V_n(r)$ is a useful pedagogical device, and studying the formula reveals some interesting properties. For example, $V_n(1)$ increases for $0\leqslant n\leqslant 5$ and decreases for $5\leqslant n<\infty$. Its maximum value is $V_5(1)=8\pi^2/15$, and $\lim_{n\to\infty}V_n(1)=0$. In fact $\sum_{n=0}^{\infty}V_n(r)$ converges for all $r>0$, and $\sum_{n=0}^{\infty}V_{2n}(r)=e^{\pi r^2}$. Thus, for fixed r, $V_n(r)$ tends to zero as n tends to infinity.

We derive the volume formula in three ways. The first and second methods use cross-sections and Fubini's theorem, and the third uses a polar-coordinate transformation and some simple properties of determinants.

The asymptotic properties already mentioned and some others are then developed.