# 2019 William Lowell Putnam Mathematical Competition Problems

**A1:** Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where $A, B$, and $C$ are nonnegative integers.

**A2:** In the triangle $\Delta ABC$, let $G$ be the centroid, and let $I$ be the center of the inscribed circle. Let $\alpha$ and $\beta$ be the angles at the vertices $A$ and $B$, respectively. Suppose that the segment $IG$ is parallel to $AB$ and that $\beta = 2\tan^{-1}(1/3)$. Find $\alpha$.

**A3:** Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant $M$ such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019\,.$$

**A4:** Let $f$ be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere $S$ of radius 1, the integral of $f(x, y, z)$ over the surface of $S$ equals 0. Must $f(x, y, z)$ be identically 0?

**A5:** Let $p$ be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo $p$. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p\,.$$

Find the greatest nonnegative integer $n$ such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

**A6:** Let $g$ be a real-valued function that is continuous on the closed interval $[0, 1]$ and twice differentiable on the open interval $(0, 1)$. Suppose that for some real number $r > 1$,

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} = 0\,.$$

Prove that either

$$\lim_{x \to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x \to 0^+} x^r |g''(x)| = \infty\,.$$

**B1:** Denote by $\mathbb{Z}^2$ the set of all points $(x, y)$ in the plane with integer coordinates. For each integer $n \geq 0$, let $P_n$ be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points $(x, y)$ such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of $n$, the number of four-point subsets of $P_n$ whose elements are the vertices of a square.

**B2:** For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right) \cos^2\left(\frac{k\pi}{2n}\right)} .$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3} .$$

**B3:** Let $Q$ be an $n$-by-$n$ real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where $I$ is the $n$-by-$n$ identity matrix. Show that if 1 is not an eigenvalue of $Q$, then 1 is an eigenvalue of $PQ$.

**B4:** Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$x f_x + y f_y = xy \ln(xy) ,$$
$$x^2 f_{xx} + y^2 f_{yy} = xy .$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s\geq 1} \Big( f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) \Big) .$$

Determine $m(f)$, and show that it is independent of the choice of $f$.

**B5:** Let $F_m$ be the $m$th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n + 1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers $j$ and $k$ such that $p(2019) = F_j - F_k$.

**B6:** Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If $p$ is in $S$, then none of the neighbors of $p$ is in $S$.

(2) If $p \in \mathbb{Z}^n$ is not in $S$, then exactly one of the neighbors of $p$ is in $S$.