The 84th William Lowell Putnam Mathematical Competition 2023

B1 Consider an m-by-n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

B2 For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

B3 A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \ldots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \ldots, i_k$ such that $X_{i_1}, X_{i_2}, \ldots X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

B4 For a nonnegative integer n and a strictly increasing sequence of real numbers $t_0, t_1, \ldots, t_n$, let $f(t)$ be the corresponding real-valued function defined for $t \geq t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \geq t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \ldots, t_n$;

(b) $f(t_0) = 1/2$;

(c) $\lim_{t \to t_k^+} f'(t) = 0$ for $0 \leq k \leq n$;

(d) For $0 \leq k \leq n-1$, we have $f''(t) = k+1$ when $t_k < t < t_{k+1}$, and $f''(t) = n+1$ when $t > t_n$.

Considering all choices of n and $t_0, t_1, \ldots, t_n$ such that $t_k \geq t_{k-1} + 1$ for $1 \leq k \leq n$, what is the least possible value of T for which $f(t_0 + T) = 2023$?

B5 Determine which positive integers n have the following property: For all integers m that are relatively prime to n, there exists a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\pi(\pi(k)) \equiv mk \pmod{n}$ for all $k \in \{1, 2, \ldots, n\}$.

B6 Let n be a positive integer. For i and j in $\{1, 2, \ldots, n\}$, let $s(i, j)$ be the number of pairs (a, b) of nonnegative integers satisfying $ai + bj = n$. Let S be the n-by-n matrix whose (i, j)-entry is $s(i, j)$.

For example, when $n = 5$, we have $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$.

Compute the determinant of S.