

Number Theory

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1 Student audience

Number theory is an attractive way to combine deep mathematics with familiar concrete objects and is thus an important course for all mathematics students: for “straight” mathematics majors, for pre-service high school teachers, and for students who are preparing to go to graduate school. In addition, it attracts some engineering and computer science students, especially those with an interest in understanding encryption.

The course can be taught at various levels. Students have been studying the natural numbers all of their lives, so proofs of the basic properties of the real numbers are an excellent vehicle for introducing students to proof. Because subtleties crop up fairly quickly, Number Theory is a good transition course in serving to turn a student’s attitude about mathematics classes from the test-oriented calculus mode of learning towards a more mature concept-oriented approach; thus some institutions use their Number Theory course as a “transition to proofs” course for their majors. Other introductory Number Theory courses assume that students have some facility with proof and serve as an introduction to more advanced mathematics.

2 History of the course

The teaching of number theory seems to be fairly consistent over time with a couple of exceptions. In terms of content, the use of number theory in modern cryptography has led to the appearance of relevant applications towards the end of some courses. Furthermore, the increasing accessibility of computer algebra systems has led to some rethinking of the old emphasis on computational short-cuts involving modular arithmetic. (On the other hand, these can now be taught as a means of explaining algorithms used by computers.)

3 Cognitive goals

Analytical and critical thinking are at the core of any Number Theory course. Students' familiarity with the positive integers means that they can begin to apply the results of the course quickly. At the same time, the existence of apparently rare but easily verified counterexamples (e.g. Fermat primes, pseudoprimes) forces the student to acknowledge that patterns may break down. Number theory is also famous for having a large number of problems whose difficulty is, shall we say, not obvious to discern on first reading. (This is an issue for anyone teaching the course, or using a particular textbook, for the first time.)

Creativity in Number Theory courses is closely tied to problem-solving and making conjectures. Students should be encouraged to look for patterns and make conjectures and try to verify those conjectures. It is important, of course, to understand that easily made conjectures may be very difficult to verify, even if they are true. Moreover, smart people have been doing number theory for centuries and it is not likely that a beginning student will be able to find something which is simultaneously new and interesting. (This issue of course faces professional number theorists even more so!)

4 Mathematical outcomes

All introductory Number Theory courses seem to cover the following core topics:

- *Divisibility*: the division algorithm, greatest common divisors and least common multiples, the Euclidean algorithm, prime numbers and their properties, the fundamental theorem of arithmetic
- *Congruences*: basic properties, complete and reduced residue systems, linear congruences, Chinese remainder theorem, Wilson's theorem, Fermat's theorem
- *Arithmetic Functions*: multiplicative functions, the Euler phi function, the sum-of-divisors function

Beyond these ideas, Number Theory courses tend to fall into two main types, which affects what additional topics are studied in the course:

Type I– A course that focuses largely on developing students’ proof-writing skills. Within this type there are two subtypes:

Ia– The course is the students’ first serious introduction to proof-writing; such a course usually has calculus and/or linear algebra as its prerequisites.

Additional topics: methods of proof (direct proof, proof by contrapositive, and proof by induction) and they help students learn to create their own proofs.

Ib– The course has introduction to proof-writing as a prerequisite but focuses on further developing those skills; in this way the course is viewed as a preparation for abstract algebra and/or real analysis, or taught in parallel with them.

Additional topics: quadratic residues, the law of quadratic reciprocity, and primitive roots.

Type II– A course that focuses largely on gaining greater depth in mathematics, with Abstract Algebra I as a prerequisite.

Additional topics: These courses have varying syllabi that depend on the instructor’s expertise and interests. Some seem to delve further into algebraic Number Theory; others develop topics related to analytic Number Theory.

Additional topics: applications of congruences (cryptography, primality testing), distribution of primes, perfect numbers, continued fractions, Gaussian integers, Diophantine equations. *More advanced topics:* number-theoretic functions, p -adic numbers, Bernoulli numbers and Bernoulli polynomials.

As a future issue, the nature of computation in the course is unresolved. There are three natural modes: pen and paper, using computer algebra systems and writing simple programs directly. Computers can be used to test conjectures generated in class, or to look at certain types of numbers statistically. For example is it common or uncommon for a natural number to be expressible as a sum of two squares? Increasingly, computer algebra systems are being used to implement the RSA encryption process or to carry out some other application requiring lengthy computations.

5 Trolling for information and ideas

With the permission of Gavin LaRose, webmaster for the Project NeXT mailing Lists, the chair of the committee asked members of the six Project NeXT lists whether they would be willing to answer the following three questions:

1. How is undergraduate number theory taught in your institution? (That is: syllabus, textbook, goals, prerequisite, target audience, technology used, writing requirements, etc. Specific syllabi are welcome!)
2. How does your undergraduate Number Theory course fit into your department's expectations for the undergraduate math major?
3. In an ideal educational environment for your institution, how would the Number Theory course differ from the one you have now?

The Project NExT survey led to about 30 responses, some from colleagues of the NExT-ers who had been contacted. ¹ The committee would like to acknowledge the help of those who responded. Here is an alphabetical list; those who gave particularly detailed responses are noted by an asterisk.

Justin Brown* (Olivet Nazarene), Duff Campbell* (Hendrix), David Cox* (Anherst), Harold Diamond (UIUC), Geoffrey Dietz* (Gannon), Karrolyne Fogel* (Cal Lutheran), Chris French* (Grinnell), Darren Glass* (Gettysburg), Bonnie Gold* (Monmouth), Joshua Holden* (Rose-Hulman), Marty Isaacs* (Wisconsin), Paul Jenkins* (BYU), Eric Kahn* (Bloomsburg), Erika King* (Hobart and William Smith), Matt Koetz* (Nazareth), Carl Lutzer (RIT), Ted Mahavier (Lamar), David Murphy* (Hillsdale), Jennifer Paulhus* (Grinnell), Tommy Ratliff (Wheaton), Thomas Roby* (Connecticut), Doug Shaw* (UNI), Thomas Sibley (St. John's), Michael Starbird* (Texas), Jeffrey Stopples* (UCSB), Chris Storm* (Adelphi), Wayne Tarrant (Wingate), Steve Ullom* (UIUC), Carolyn Yackel* (Mercer), Huiya Yan (Wisconsin-La Crosse), Andrew Yang* (Dartmouth), Nicholas Zoller* (Southern Nazarene).

6 Dreaming

The most persistent dream among our respondents was for math departments and undergraduate programs to be large enough that Number Theory courses

¹The chair sent a similar, more narrowly focused, email to the faculty on the number theory mailing list at UIUC. He did not receive much useful material from this list.

could be offered more often and with a greater variety.

Instructors often mentioned the wide range of differences in background for students from their own program. An advanced undergraduate mathematics course would be richer if one could assume a stronger background in other courses. In particular, if students know about groups, rings and fields already, much of the basics of modular arithmetic could be just briefly reviewed. Several people expressed a desire to get far enough to talk about elliptic curves.

7 Bibliography

What follows is a list of the textbooks used by our highly unrandom and self-selected group of Project NExT alumni mentioned above. The integers indicate the number of times a text was listed either as primary or supplemental by one of the respondents.

Remark: *The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. Please note that some of the books listed below were written by respondents of the poll.*

1. Agnew, Jeanne, *Explorations in Number Theory*. Brooks-Cole Publishing, Co, 1972. (1)
2. Anderson, James A and James M. Bell, *Number Theory with Applications*, Prentice Hall, 1997. (1)
3. Andrews, George E, *Number Theory*, Courier Corporation, 2012. (2)
4. Burton, David, *Elementary Number Theory*, McGraw Hill Education, 2010. (4)
5. Dudley, Underwood, *A Guide to Elementary Number Theory*, MAA Publications, 2009. (4)
6. Edwards, Harold M. *Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory*, Springer Science and Business Media, 2000. (1)

7. Jones, Gareth A. and Josephine M. Jones, *Elementary Number Theory*, Springer Science and Business Media, 2012. (2)
8. Marshall, David C., Edward Odell, and Michael Starbird, *Number Theory through Inquiry*, MAA Textbooks, 2007. (4)

Note: this text is written to support an inquiry-based approach that elicits more intense student involvement in finding the truths of Number Theory at the expense of covering less material.

9. Niven, Ivan, Herbert S. Zuckerman, and Hugh L. Montgomery, *An Introduction to the Theory of Numbers*, Wiley, 1991. (2)
10. Pommersheim, James, Tim Marks, and Erica Flapan, *Number Theory: A Lively Introduction with Proofs, Applications, and Stories*, Wiley, 2010. (3)
11. Rosen, Kenneth H., *Elementary Number Theory and Its Applications*, Pearson, 2010. (3)
12. Silverman, Joseph H., *A Friendly Introduction to Number Theory*, 4th Edition. Pearson, 2013. (3)
13. Strayer, James K., *Elementary Number Theory*, Waveland Press, 2001. (1)
14. Tattersall, James J., *Elementary Number Theory in Nine Chapters*, Cambridge University Press, 2005. (1)

One course gave directed readings in the book

15. Singh, Simon, *Fermat's Enigma*, Paw Prints, 2009.

It is increasingly the case that enterprising instructors (and enterprising students) can piece together significant amounts of supplemental text, software and video on the web.