Carl B. Allendoerfer Awards

The Carl B. Allendoerfer Award, established in 1976, is made to authors of expository articles published in *Mathematics Magazine*. Carl B. Allendoerfer, a distinguished mathematician at the University of Washington, served as president of the Mathematical Association of America, 1959–60.

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“Invisible Knots and Rainbow Rings: Knots Not Determined by Their Determinants,” *Mathematics Magazine*, 93:1, 4–18. 10.1080/0025570X.2020.1685320

This article has it all. Fun and games for all ages? Check. Careful, logical build-up of terminology and mathematical results? Check. A writing style that is conversational yet mathematically precise? Check. Fascinating connections across seemingly disparate areas of study within mathematics? Check. Well-chosen examples that answer reader questions just before we think to ask them? Check. Drawings to show those examples and build our intuition and understanding? Check. And finally: an ending that encourages the reader to do more, simultaneously issuing a challenge while also providing support for how to proceed? Check!

“Invisible Knots and Rainbow Rings: Knots Not Determined by Their Determinants” charts a delightful path that begins with Möbius strips, instantly engaging readers with ideas for Möbius strip variations to try themselves. We quickly learn these extensions are called paradromic rings, and there are patterns based on how many half twists we make, and based on whether we bisect the strip or cut it into even more sections. There is vocabulary for all this, starting with terms like knots and links, then describing the intriguingly named invisible knots and rainbow rings of the article’s title.

Paradromic ring diagrams may be colored, similar to how graphs may be colored, and this is where the mathematical details and connections in this paper shine. Coloring requirements are first stated as two seemingly simple conditions, and the diagrams for some of the easier-to-visualize cases suggest it may be straightforward to decide colorability. However, readers likely guess that more complicated cases exist! This article talks us through these cases by providing multiple ways to determine the colorability of paradromic rings. We learn that drawings, besides building our intuition, also show a consistent way to re-draw paradromic rings to better visualize and count all their crossings. We revisit the single equation of the original two simple conditions, expanding this equation into a matrix-vector equation that is generalizable to examples involving many crossings. The eigenvalues of this matrix form the crux of proofs about colorability possibilities. We then learn about torus links, which partition paradromic rings into cases: some are torus links, and some are not. The authors build their case engagingly and convincingly, using all these ideas, and culminating with a complete characterization of the colorability of paradromic rings. Though they have proved all their results, they leave one proof out of the article as a temptation for readers. We are left with an invigorating call to learn more and complete the proof, as well as with specific suggestions for books to read and topics to focus on. With this guidance, we as readers believe fully that we can progress on this work.

The authors have introduced us to their topic in a way that feels natural to anyone who has ever played with a Möbius strip. They then lead us through coloring, equations, knots, linear algebra, and proofs, all while sounding like we are chatting with a friend. They leave us inspired to try more and persuaded that we can very definitely make progress. Throughout, we remain engaged and find new connections in mathematics.

Response

It is an honor to receive this award. Getting this paper published was a long and winding road and we're grateful to the many people who helped along the way. Thomas would like to dedicate this award to the many students who worked with him, especially those who worked hard only to find that the results could not be published. We also want to encourage those who have something to say to not be discouraged and keep plugging away at it. In the hopes that it can be an inspiration, let us tell you that it took more than a decade between doing this research and getting it published.

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The first three authors were undergraduates at the time while Dan Sours is a high school teacher.

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Biosgraphical Sketch

James Godzik completed a Bachelor's degree at UC, Berkeley and a Master's in teaching mathematics at CSU, Fullerton.

Nancy Ho received a BA in mathematics from Mills College in 2006 and a PhD in mathematics from University of Oklahoma in 2015. She is currently working as a software engineer with Tapestry Solutions.

Jennifer Jones was an undergraduate at Colorado State University at the time of this research.

Thomas W. Mattman received a PhD in mathematics from McGill University. His mathematical interests include knot theory and graph theory and he enjoys supervising research by undergraduate students and teachers. In real life, he's a devoted husband to Shigemi and a doting father to Saya and Aki; who provide the comic relief.

Daniel M. Sours received a BS in mathematics (1985) and engineering (1981) and a MS in mathematics education (2004) from California State University, Chico. He has taught at Chico High School in Chico California since 1987 and has also served as adjunct faculty at California State University, Chico and Butte Community College. He adores his wonderful wife Mary and loves relaxing with her on the ocean in Little River, California.