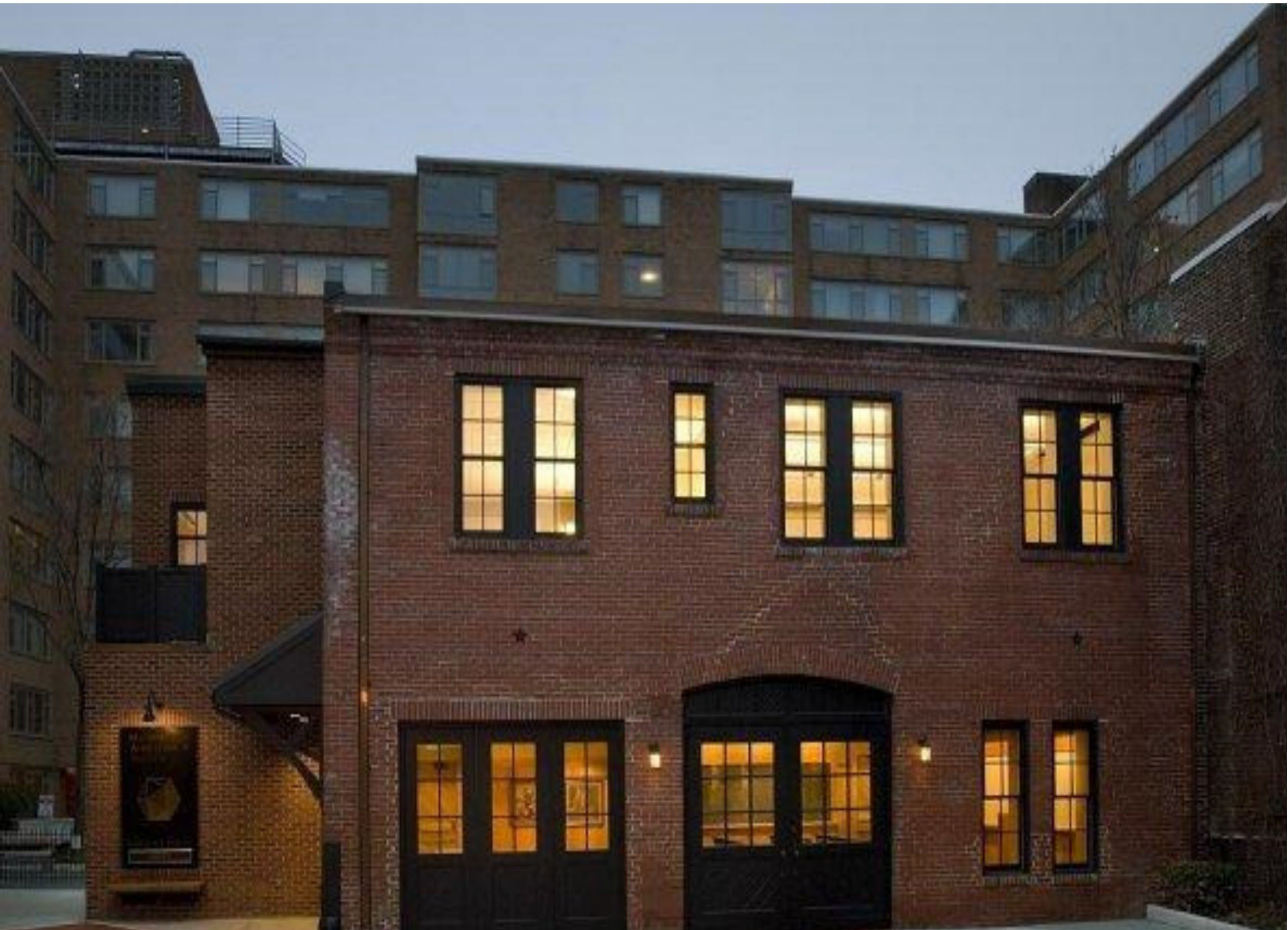

Abstract Algebra

A proposed update for the 2015 CUPM Guide to Undergraduate Programs in Mathematics Sciences

July 25, 2022

MATHEMATICAL ASSOCIATION OF AMERICA

Abstract Algebra

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Overview

This is a first course (or sequence) in abstract algebra, typically the study of groups, rings, and fields. As departments typically have different needs for abstract algebra, we give examples of four different versions: a single course for mathematics majors intending to continue on to either advanced study in mathematics or to an industry job using mathematics; a sequel course that may or may not be required of those students, but of particular need for students intending to continue on to a Ph.D. program; a single course intended for students who are considering being future secondary (or middle) school teachers; or a single course that must meet the needs of all students.

Student Audience

As mentioned above, the student audience for the three courses (and the fourth follow-on course) are typically mathematics or mathematics education students. The course(s) would typically be taken by junior or senior students majoring (or minoring) in mathematics.

Cognitive Goals Addressed

Students develop effective thinking and communication skills.

Awareness of logical coherence: Students should be aware of the logical flow of ideas in the development of the various theorems of abstract algebra. More specifically, they should understand that results depend on earlier results, and are not merely random facts. *Examples:* In group theory, students should see the connections between cosets, Lagrange's theorem, normality and factor groups. In ring theory, students should be aware of which facts about the ring of polynomials over a field are consequences of the division algorithm.

Understanding formal definitions: Students should see the importance of precise formal definitions, including the definitions of groups, rings and fields by sets of axioms. They should appreciate the fact that these axiomatic systems apply to many different concrete objects. Students should also understand and be able to apply definitions of algebraic objects such as subgroups, homomorphisms and ideals, and of adjectives such as "abelian" and "normal." Students should be able to produce examples of these concepts and to solve problems concerning them. *Examples:* Students should be able to provide examples of both commutative rings and non-commutative rings, of both subrings that are ideals and subgroups that are not ideals, of both normal subgroups and subgroups that are not normal, of isomorphisms and homomorphisms are not isomorphisms, of abelian groups and non-abelian groups of order 10, etc.

Communication: Students should be able to communicate their mathematical ideas clearly, completely, and concisely, both in writing and in speaking. *Examples:* Some assignments should require carefully written paragraphs, and they should be graded at least in part on the clarity and coherence of the writing. Students should have opportunities to present their ideas orally in class.

Conjectures and justifications: Students should be able to recognize patterns and formulate conjectures about those patterns, and they should be able to create examples to test their conjectures, and perhaps find proofs or counterexamples. Students should also be able to read and critique the arguments of others. *Example:* In the study of finite groups, but before Lagrange's theorem is introduced, students might make conjectures about possible orders of elements.

Students learn to link applications and theory.

Students should be able to describe connections between abstract algebra and other mathematics courses they have taken, and they should be able to apply algebra to solve problems in other areas of mathematics and in other disciplines. Students preparing for teaching secondary school mathematics should be able to explain the links between the concepts of abstract algebra and corresponding concepts in the high school mathematics curriculum.

Examples:

- Connections to linear algebra: Students should recognize that vector spaces are additive groups and that linear transformations are group homomorphisms. Students should see that the determinant map is a homomorphism from GL(n, R) to the multiplicative group of nonzero real numbers.
- Connections to calculus: Students should see that polynomials over the real numbers (or real-valued functions) form a ring and that differentiation is an additive group homomorphism but not a ring homomorphism.
- Connections to the high school curriculum: Students should see the complex numbers as isomorphic to the 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. For other examples, see the description of Algebra E in Section 5.

Students learn to use technological tools.

While Tex/LaTeX are not necessarily covered, this is a great class to have students learn, practice, or improve these skills. Similarly many other programs might be useful. Departments should consider using such other programs in this class and in the future may wish to require new technologies. There may be specific topics where students would benefit from technology and programs (for example computer algebra systems such as Sage, Magma, and GAP might prove useful). Consequently departments are encouraged to keep abreast of the changes in technology and whether they will be applicable to improving the abstract algebra experience.

Students develop mathematical independence and experience open-ended inquiry.

Such goals include helping students build confidence at performing mathematics and encouraging them to take ownership of their own learning. Learning to persevere is another important goal for students of mathematics. We note that many teachers have had success with an active learning or inquiry-based approach.

Specific Student Learning Outcomes

Learning outcomes depend on the version of the course being taught (see Possible Syllabi section).

1. Students will be able to demonstrate understanding of the basic definitions and properties of groups, rings, and/or fields. Students will be able to state basic definitions, give basic examples of each, and be able to create examples satisfying certain properties. Students will additionally be able to evaluate whether or not an object satisfies a definition.

2. Students will be able to demonstrate an understanding of the basic structural properties of groups. Students should be able to define basic properties of subgroups, normal subgroups, factor groups, cyclic groups, permutation groups, group homomorphisms, and group isomorphisms. Students will be able to state and use Cauchy's theorem, Lagrange's theorem, and the Fundamental Isomorphism Theorem for groups. Students will be able to solve problems involving these concepts and be able to explain key aspects of the proofs of these theorems.
3. Students will be able to demonstrate an understanding of the basic structural properties of rings and fields. Students should be able to define basic properties of subrings, ideals, integral domains, rational, real and complex fields, polynomial rings, matrix rings, and ring homomorphisms. Students will be able to state and use the Fundamental Isomorphism Theorem for rings and the theorem that a commutative ring with 1 modulo a maximal ideal is a field (or modulo a prime ideal is an integral domain). Students will be able to solve problems involving these concepts and being able to explain key aspects of proofs of important theorems.
4. Students will be able to show an improvement (from the beginning of the course) in building intuition about abstractly defined concepts. Students will be able to easily find examples that satisfy, or alternatively don't satisfy, a given hypothesis.
5. Students will improve (from the beginning of the course) their proof writing skills and their mathematical writing skills in general. (The specific statement of this learning outcome will depend on where the algebra course comes in the specific department's curriculum).
6. Students will improve (from the beginning of the course) their oral communication of mathematics. As with writing, the specific statement of this outcome (and whether it should be included at all) will depend on where the course comes in the department's curriculum. However, in a course for future teachers, we highly recommend this outcome be included (and strongly recommend for others).
7. Students will be able to use an application of abstract algebra in other contexts. (The specific statement of this learning outcome will depend on the specific department's curriculum).
8. Students will be able to recognize and create valid mathematical arguments in abstract algebra. Students should also be able to recognize and critique incorrect arguments.

Core Topics

Groups:

Essential Topics

1. Definitions and examples of groups and subgroups. Examples should include but not be limited to groups of rotations and reflections of planar figures and rotations of 3-dimensional objects, symmetric groups, integers modulo *n* with respect to addition and unit groups of integers modulo n with respect to multiplication, invertible 2×2 real matrices under multiplication.
2. Cyclic groups and their subgroups, and the orders of elements.
3. Symmetric groups, cycle notation, parity of a permutation and the alternating group.
4. Isomorphisms.
5. Cosets and Lagrange's Theorem, the falsity of the converse of Lagrange's Theorem, and the statement of the Sylow existence theorem.
6. Normal subgroups and factor/quotient groups, conjugates of a subgroup and of an element.
7. Homomorphisms and the Fundamental Isomorphism Theorem for groups.

Optional Topics or topics that might be in an Algebra for future teachers course (see the section on possible syllabi)

1. Cayley's Theorem.
2. Cauchy's Theorem
3. The proof of the Sylow existence theorem.
4. Group actions.

5. External direct products.
6. Statement of the structure theorem for finite abelian groups.
7. Planar transformation groups.
8. Orbit Stabilizer Theorem.
9. Class Equation.
10. Burnside's orbit counting formula.

Rings/Fields:

Essential Topics
1. Definitions and examples of rings and subrings. Examples should include but not be limited to the integers; the integers modulo *n* (including the fact that if *n* is prime the ring is a field); the rational, real, and complex fields; polynomial rings; and matrix rings.
2. Ideals, including principal ideals.
3. Factor/Quotient rings including the construction of the complex numbers as the factor ring $R[x]/(x^2+1)$.
4. Integral domains and principal ideal domains.
5. Isomorphisms.
6. Irreducible Polynomials over Q.
7. Homomorphisms, and the Fundamental Isomorphism Theorem for rings.

Optional Topics or topics that might be in an Algebra for future teachers course (see the section on possible syllabi)
1. Finite fields of non-prime order.
2. Gaussian integers.
3. Maximal and prime ideals.
4. DeMoivre's Theorem.
5. The division algorithm (used to prove the integers are a principle ideal domain).
6. Statement of the Fundamental Theorem of Algebra.
7. The Impossibility Theorems (Impossibility of trisection of an angle, squaring the circle, and doubling the cube with straightedge and compass).
8. The factor theorem/remainder theorem (that f(a) is the remainder when f(x) is divided by (x-a)).
9. The rational root theorem.
10. The solution of the depressed cubic and the development of the complex numbers.
11. Minimal and characteristic polynomials and the Cayley-Hamilton Theorem.
12. Nilpotent Operators and Jordan Canonical Form.
13. Algebraic Extension Fields.
14. Splitting fields.
15. Existence and uniqueness of finite fields.
16. Galois theory.

Prerequisite Skills and Knowledge

Abstract algebra can fit in many different places in the curriculum of a mathematics department. Students should have a background in elementary linear algebra (knowledge of real vector spaces and 2×2 and 3×3 matrices over the real and complex numbers at minimum). Abstract algebra can function as a first "proofs" course, but even in that case, students should have an understanding of mathematical argumentation, where at the minimum they can write a solution to a calculus or linear algebra problem in complete sentences. We note that the depth of the abstract algebra course and expectations of students should depend on where the course comes in a department's curriculum.

Possible Syllabi

We note that there are four distinct types of abstract algebra courses (or sequences) that could be offered by a department depending on the size and resources of the college/university. We will call these four courses: Algebra A, Algebra AB, Algebra E, and Algebra EAB.

Algebra A is a single course in abstract algebra for math majors including but not limited to students intending to teach high school. Algebra AB is a two-course sequence in abstract algebra, primarily taken by mathematics majors. Algebra E is a single course in abstract algebra taken by future high school teachers. Algebra EAB is a sequence in abstract algebra for departments whose resources limit them to one sequence with future teachers only taking a single course. These four courses (or sequences) are general in nature, and allow for adapting to local circumstances.

We further note that there are two standard ways of approaching an abstract algebra course: groups first or rings first. Each has its own advantages. The groups-first approach has the advantage that groups are the simpler object (one operation and fewer axioms). Those who favor the groups-first approach appreciate that the students' lack of familiarity assists them in seeing proof as a way of learning what is true about an object. The rings-first approach has the advantage that students' familiarity with the integers and the rational numbers gives the students many examples. Moreover, this approach more closely follows the historical development of abstract algebra. We note that in either approach, we strongly recommend that the parallelism between the definitions and theorems for groups and the definitions and theorems for rings be emphasized.

We strongly suggest that departments consider a wealth of factors in determining whether to offer a course or a sequence and whether to choose to focus on the topics from the E versions. In particular, departments should look at the ratio of students who plan on going into teaching versus those planning to go into the job market versus those planning to attend graduate programs in mathematics. Most importantly, departments should consider the algebra course/sequence's place in their curriculum. Is it a first proofs course, a capstone course, or somewhere in between? Departments should look at their curriculum map and where students are in the mastery of their cognitive goals under their standard curriculum plan. Just like one should not see algebra as a bunch of disconnected pieces, a department should not see its curriculum as a set of disconnected courses.

Algebra A:
In Algebra A, we recommend the instructor cover all of the 7 essential topics from groups and the 7 essential topics from rings/fields. For groups, we note that there are several unifying themes that can be used in covering these topics, including focusing on permutation groups, focusing on group actions, or focusing on multiplication tables and their block format. One option for covering additional topics that might be particularly useful in the case of a diverse student audience would be to assign projects to students from the additional topics based on student interest or their future interests.

Algebra AB:
In this sequence, we recommend that in addition to the topics from Algebra A, the instructor choose 7-14 topics from the optional topics for groups and rings (Group optional 1-5 and 7-9 and Ring optional 1-3 and 10-16). If most students take the entire sequence, then we think that the ordering of the topics between the first and second courses can be done as the instructor sees fit over the year. On the other hand, if a substantial number of the students will only take the first course, we strongly recommend that most of the essential topics from groups and rings remain in the first course.

Algebra E:

In this course meant for future teachers of mathematics, we feel that a few of the essential topics can be left out so as to include optional topics 4-9 from Ring theory and optional topic 6 from group theory (if groups are emphasized). For such a course with a rings-first approach, a potential ordering of topics might be as follows:

- The ring axioms and the integers.
- The division algorithm for the integers.
- The field axioms and the rational, real, and complex numbers.
- DeMoivre's theorem and exponential notation for complex numbers.
- Integers modulo n as rings and fields.
- Integer divisibility theorems.
- Matrix rings.
- Polynomial rings.
- The division algorithm for polynomials.
- The connection between polynomial division and base 10 division or more generally between polynomial arithmetic and base 10 arithmetic.
- Ideals (with key examples over the integers and polynomial rings.)
- Factor rings (Integers modulo n, polynomials modulo (x^2+1).)
- Isomorphisms.
- Homomorphisms and the Fundamental Homomorphism Theorem.
- The Remainder Theorem and the Rational Root Theorem.
- Definition of a group.
- Integers mod n, 2×2 matrix groups.
- Transformation groups on the plane.
- Essential group theory topics as time permits.

We further recommend that anyone teaching such a course should familiarize themselves with the MET 2 report (http://www.cbmsweb.org/archive/MET2/met2.pdf) and its recommendations.

Algebra EAB:
Algebra EAB is a sequence where future teachers take only the first course in the sequence. In the first course of this sequence, we recommend covering most of the topics from the Algebra E course, and then assuring that all of the 7 essential topics from groups and the 7 essential topics from rings/fields are covered. As in Algebra AB, we recommend that the instructor choose among the additional topics to create a cohesive second course/sequence.

Student Assessments

We list below the key cognitive learning outcomes and types of assignments that we recommend.

Students develop effective thinking and communication skills*: Awareness of logical coherence, Understanding formal definitions, Communication, Conjectures and justifications.*

- Traditional homework exercises in texts are a primary way to assess in this area.
- Poster presentations,
- Having students evaluate proofs written by other students
- Other active learning strategies like group assignments

Students learn to link applications and theory.

- Projects/posters linking abstract algebra to other areas/applications.

- Assigning reports on papers from *Math Horizons*, *NCTM journals,* or other places that have papers on applications accessible to students.

Students learn to use technological tools.
If appropriate to the class and curriculum:

- Turning in at least one assignment in LaTeX.
- Using Sage, Python, Mathematica, or computer algebra system to investigate a conjecture.

Students develop mathematical independence and experience open-ended inquiry.

- Open-ended student projects

References and Resources

We list below some books that might be considered as possible texts for an undergraduate Abstract Algebra course, and we list separately books that instructors (or students) might want to consult to gain further insight into algebra, its history and its pedagogy. Of course, the line between these two categories of books is not clearly defined. Please note that some of the books listed were written by the authors of this report.

Textbooks

> Remark: The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of Abstract Algebra courses.

1. Artin, Michael, Algebra,2nd edition. Pearson, 2011. (A textbook for advanced undergraduates)
2. Birkhoff, Garrett and MacLane, Saunders, A Survey of Modern Algebra, A. K. Peters, 1998. (This is a classic, originally published in 1941. It treats rings first.)
3. Carter, Nathan, Visual Group Theory, MAA, 2009. (Many diagrams, but group theory only.)
4. Cox, David; Little, John and O'Shea, Donal, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra Springer, 2nd edition, Corrected printing, 2005. (This might be suitable for a second Abstract Algebra course different from our Algebra B.)
5. Cuoco, Al and Rotman, Joseph J, Learning Modern Algebra: From Early Attempts to Prove Fermat's Last Theorem, MAA, 2013. (This historically organized text is suitable for prospective teachers. It also aims to show how "important themes in algebra arose from questions related to teaching.")
6. Dubinsky, Ed and Leron, Uri, Learning Abstract Algebra with ISETL. Springer, 1993. (ISETL is a programming language. This text encourages students to work with algebraic ideas.)
7. Dummit, David S. and Foote, Richard M, Abstract Algebra, 3rd edition, John Wiley & Sons, 2003. (As with Artin's book, above, this text would be suitable only for advanced undergraduates; it may be too ambitious as a text for Algebra A.)
8. Fraleigh, John B., A First Course in Abstract Algebra, 7th edition, Pearson, 2003. (A very popular undergraduate algebra textbook.)
8. Gallian, Joseph A., Contemporary Abstract Algebra, 8th edition, Cengage, 2012. (May be the most widely used undergraduate algebra text.)
9. Gilbert, L. and Gilbert J., Elements of Modern Algebra, 7th edition, Brooks/Cole, 2012. (A popular textbook that is written at a more introductory level.)
10. Goodman, F. M., Algebra: Abstract and Concrete, Prentice Hall, 2003. (Stresses symmetry. Free online.)
11. Herstein, I.N., Topics in Algebra, 2nd edition. John Wiley & Sons, 1975. (A classic. It might be used for a strong Algebra A class, and also for Algebra B.)
12. Herstein, I.N., Abstract Algebra, 3rd edition. John Wiley & Sons, 1996. (This is probably more suitable than the older Herstein text for Algebra A.)

13. Hungerford, Thomas, Abstract Algebra: An Introduction, 3nd edition, Brooks/Cole, 2014. (One of the few textbooks that uses a rings-first approach.)
14. James, Gordon and Liebeck, Martin, Representations and Characters of Groups, Cambridge Mathematical Textbooks, Cambridge University Press, 1993. (Suitable for a "topics" course following Algebra A.)
15. Judson, Thomas W., Abstract Algebra: Theory and Applications, 1997 (revised 2011). (The text of the book is available free online here and available in Sage worksheet format here.)
16. Rotman, Joseph J., A First Course in Abstract Algebra, 3rd edition, Pearson, 2005. (This book does some commutative algebra as well as topics from linear algebra, including canonical forms.)
17. Schifrin, Theodore, Abstract Algebra: a Geometric Approach, Pearson, 1995. (Its genesis was a series of algebra workshops for in-service teachers. The result is a text that would be particularly suitable for Algebra E.)
18. Solomon, Ronald, Abstract Algebra, Pure and Applied Undergraduate Texts 9, AMS, 2003. (This text takes a novel approach, including topics not often covered in undergraduate texts. It emphasizes the historical development of the ideas and is aimed at prospective high school teachers.)
19. Shahriari, Shahriar, Algebra in Action: A Course in Groups, Rings, and Fields (Pure and Applied Undergraduate Texts), American Mathematical Society, 2017. (A text that introduces group actions early and uses actions as a unifying theme.)
20. Stillwell, John, Naive Lie Theory, Undergraduate Texts in Mathematics, Springer- Verlag, 2008. (Suitable for a "topics" course following Algebra A.)
21. Tapp, Kristopher, Matrix Groups for Undergraduates, Student Mathematical Library 29, American Mathematical Society, 2005. (Suitable for a "topics" course following Algebra A.)

Other relevant books

1. Bashmakova, I.G., and Smirnova, G. S., The Beginnings and Evolution of Algebra, MAA, 2000.
2. Conference Board of the Mathematical Sciences. The Mathematical Education of Teachers, Volume 11 in the CBMS series Issues in Education, AMS and MAA, 2001.
3. Conference Board of the Mathematical Sciences. The Mathematical Education of Teachers II, Volume 17 in the CBMS series Issues in Education, AMS and MAA, 2012. (Available at www.cbmsweb.org.)
4. Cooke, Roger, Classical Algebra: Its Nature, Origins, and Uses, John Wiley & Sons, 2008.
5. Isaacs, I. Martin, Algebra: A Graduate Course, AMS Graduate Studies in Mathematics, 1994; originally published by Brooks/Cole. (This advanced text covers most of the topics in our syllabi thoroughly. It is a good reference for instructors and advanced students.)
6. Kleiner, Israel, A History of Abstract Algebra, Birkhauser, 2007.
7. Parker, Ellen Maycock, Laboratory Experiences in Group Theory: A Manual to Be Used With Exploring Small Groups. MAA, 1996.
8. Sultan, A. and Artzt, A., The Mathematics that Every Secondary School Math Teacher Needs to Know, Routledge, Taylor & Francis, 2011.
9. Usiskin, Z., Peressini, A. L., Marchisotto, E. and Stanley, D., Mathematics for High-School Teachers — An Advanced Perspective, Pearson, 2003.
10. van der Waarden, B. L., Modern Algebra, volumes 1 and 2, Springer, 1991. (Originally published in German in 1930. A true classic, but not suitable as an undergraduate text.)
11. Wussing, Hans, The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory, Dover, 2007.