

A Quotient Rule Integration by Parts Formula

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In a recent calculus course, I introduced the technique of Integration by Parts as an integration rule corresponding to the Product Rule for differentiation. I showed my students the standard derivation of the Integration by Parts formula as presented in [1]:

By the Product Rule, if $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Integrating on both sides of this equation,

$$\int [f(x)g'(x) + g(x)f'(x)] dx = f(x)g(x),$$

which may be rearranged to obtain

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

Letting $U = f(x)$ and $V = g(x)$ and observing that $dU = f'(x) dx$ and $dV = g'(x) dx$, we obtain the familiar Integration by Parts formula

$$\int U dV = UV - \int V dU. \quad (1)$$

My student Victor asked if we could do a similar thing with the Quotient Rule. While the other students thought this was a crazy idea, I was intrigued. Below, I derive a Quotient Rule Integration by Parts formula, apply the resulting integration formula to an example, and discuss reasons why this formula does not appear in calculus texts.

By the Quotient Rule, if $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Integrating both sides of this equation, we get

$$\frac{f(x)}{g(x)} = \int \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} dx.$$

That is,

$$\frac{f(x)}{g(x)} = \int \frac{f'(x)}{g(x)} dx - \int \frac{f(x)g'(x)}{[g(x)]^2} dx,$$

which may be rearranged to obtain

$$\int \frac{f'(x)}{g(x)} dx = \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{[g(x)]^2} dx.$$

Letting $u = g(x)$ and $v = f(x)$ and observing that $du = g'(x) dx$ and $dv = f'(x) dx$, we obtain a Quotient Rule Integration by Parts formula:

$$\int \frac{dv}{u} = \frac{v}{u} + \int \frac{v}{u^2} du. \quad (2)$$

As an application of the Quotient Rule Integration by Parts formula, consider the integral

$$\int \frac{\sin(x^{-1/2})}{x^2} dx.$$

Let

$$\begin{aligned} u &= x^{1/2}, & dv &= \frac{\sin(x^{-1/2})}{x^{3/2}} dx, \\ du &= \frac{1}{2}x^{-1/2} dx, & v &= 2 \cos(x^{-1/2}). \end{aligned}$$

Then

$$\begin{aligned} \int \frac{\sin(x^{-1/2})}{x^2} dx &= \frac{2 \cos(x^{-1/2})}{x^{1/2}} + \int \frac{2 \cos(x^{-1/2})}{x} \cdot \frac{1}{2}x^{-1/2} dx \\ &= \frac{2 \cos(x^{-1/2})}{x^{1/2}} - 2 \int \cos(x^{-1/2}) \cdot \left(-\frac{x^{-3/2}}{2}\right) dx \\ &= \frac{2 \cos(x^{-1/2})}{x^{1/2}} - 2 \sin(x^{-1/2}) + C, \end{aligned}$$

which may be easily verified as correct.

Why do we not find the Quotient Rule Integration by Parts formula in calculus texts?

First, the Quotient Rule Integration by Parts formula (2) results from applying the standard Integration by Parts formula (1) to the integral

$$\int \frac{dv}{u}$$

with

$$\begin{aligned} U &= \frac{1}{u}, & dV &= dv, \\ dU &= -\frac{1}{u^2} du, & V &= v, \end{aligned}$$

to obtain

$$\begin{aligned} \int \frac{dv}{u} &= \int U dV \\ &= UV - \int V dU \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{u} \cdot v - \int v \cdot \left(-\frac{1}{u^2}\right) du \\
&= \frac{v}{u} + \int \frac{v}{u^2} du
\end{aligned}$$

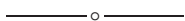
as before.

Secondly, there is the potential only for slight technical advantage in choosing formula (2) over formula (1). An identical integral will need to be computed whether we use (1) or (2). The only difference in the required differentiation and integration occurs in the computation of du versus dU . In our example, for instance, we differentiated $u = x^{1/2}$ rather than $U = x^{-1/2}$.

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Reference

1. J. Stewart, *Calculus: Early Transcendentals*, 4th ed., Brooks/Cole, 1999.



Estimating Definite Integrals

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Many definite integrals arising in practice can be difficult or impossible to evaluate in finite terms. Series expansions and numerical integration are two standard ways to deal with the situation. Another approach, primitive but often very effective, yields cruder estimates by replacing a nasty integrand with nice functions that majorize or are majorized by it. With luck and skill, the bounds achieved suffice for the task at hand. I was introduced to this method as a grad student instructor over forty years ago, when I had the good fortune to learn some innovative teaching methods from Arthur Mattuck. His supplementary notes for MIT's calculus course included a section on the estimation of definite integrals by an approach barely covered in texts back then. Many first year calculus texts of that era touched on the method in connection with comparison tests for improper integrals, but they seldom did anything with proper integrals.

The situation has improved somewhat in recent years, with prominent texts at least mentioning the basic idea within the chapter introducing the definite integral. Sometimes this is labeled the "Domination Rule" or "Comparison Property". An informal survey shows that most such books offer very few, if any, exercises in the method, usually relatively trivial ones. The texts by Edwards & Penney [1] and Stewart [4] are exceptional in providing more than a token selection of such problems. Unfortunately, their exercises are duplicated in the early transcendental versions of these two texts, thus making no use of the broader array of available functions.

Here is a sketch of the way I develop this form of estimation in my intermediate calculus course. All integrals are understood to be over a closed, bounded interval $[a, b]$