

## REFERENCES

1. R. Laatsch, Measuring the abundance of integers, this MAGAZINE **59** (1986), 84–92.
2. K. H. Rosen, *Elementary Number Theory and its Applications*, 5th ed., Pearson Addison Wesley, Boston, 2005.
3. R. F. Ryan, A simpler dense proof regarding the abundancy index, this MAGAZINE **76** (2003), 299–301.
4. P. A. Weiner, The abundancy index, a measure of perfection, this MAGAZINE **73** (2000), 307–310.

# The Associativity of the Symmetric Difference

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The symmetric difference of two sets  $A$  and  $B$  is defined by  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ . It is easy to verify that  $\Delta$  is commutative. However, associativity of  $\Delta$  is not as straightforward to establish, and usually it is given as a challenging exercise to students learning set operations (see [1, p. 32, exercise 15], [3, p. 34, exercise 2(a)], and [2, p. 18]).

In this note we provide a short proof of the associativity of  $\Delta$ . This proof is not new. A slightly different version appears in Yousefnia [4]. However, the proof is not readily accessible to anyone unfamiliar with Persian.

Consider three sets  $A$ ,  $B$ , and  $C$ . We define our universe to be  $X = A \cup B \cup C$ . For any subset  $U$  of  $X$ , define the characteristic function of  $U$  by

$$\chi_U(x) = \begin{cases} 1, & \text{if } x \in U; \\ 0, & \text{if } x \in X \setminus U. \end{cases}$$

Two subsets  $U$  and  $V$  of  $X$  are equal if and only if  $\chi_U = \chi_V$ . The following lemma is the key to our proof.

LEMMA. *For any two subsets  $U$  and  $V$  of  $X$  and for any  $x \in X$ ,*

$$\chi_{U \Delta V}(x) = (\chi_U(x) - \chi_V(x))^2 \quad (1)$$

$$= \chi_U(x) + \chi_V(x) - 2\chi_U(x)\chi_V(x). \quad (2)$$

*Proof.* Note that both sides of (1) are equal to 1 exactly when  $x$  belongs to one of  $U$  or  $V$ , but not to both. The identity (2) follows immediately from (1) and the fact that  $\chi_S^2 = \chi_S$  for any set  $S$ . ■

PROPOSITION. *Let  $A$ ,  $B$ , and  $C$  be three sets. Then*

$$(A \Delta B) \Delta C = A \Delta (B \Delta C).$$

*Proof.* From the Lemma we see that

$$\chi_{(A \Delta B) \Delta C} = \chi_{A \Delta B} + \chi_C - 2\chi_{A \Delta B}\chi_C \quad (3)$$

$$= (\chi_A + \chi_B - 2\chi_A\chi_B) + \chi_C - 2(\chi_A + \chi_B - 2\chi_A\chi_B)\chi_C$$

$$= \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C.$$

Since the last line in (3) is symmetric with respect to  $A$ ,  $B$ , and  $C$ , we conclude that

$$\chi_{(A\Delta B)\Delta C} = \chi_{(B\Delta C)\Delta A}. \quad (4)$$

But  $\Delta$  is commutative, and therefore (4) completes the proof of the Proposition. ■

By using modular arithmetic, we can make the above proof even shorter. Note that  $U = V$  if and only if  $\chi_U \equiv \chi_V \pmod{2}$ . Thus, the Lemma becomes

$$\chi_{U\Delta V} \equiv \chi_U + \chi_V \pmod{2}.$$

The associativity of the symmetric difference now follows from the associativity of addition in modular arithmetic.

## REFERENCES

1. H.B. Enderton, *Elements of set theory* (Academic Press, New York, 1977).
2. P.R. Halmos, *Naive set theory* (D. Van Nostrand, Princeton, 1960)
3. K. Kuratowski, *Introduction to Set Theory and Topology* (Pergamon Press, Oxford, 1961)
4. M. Yousefnia, A proof of associativity of symmetric difference of sets, *Roshd-e-Amuzesh-e-Riyaazi* **3** (1986), 25–26.

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