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The Associativity of the Symmetric Difference

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The symmetric difference of two sets A and B is defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. It is easy to verify that Δ is commutative. However, associativity of Δ is not as straightforward to establish, and usually it is given as a challenging exercise to students learning set operations (see [**1**, p. 32, exercise 15], [**3**, p. 34, exercise 2(a)], and [**2**, p. 18]).

In this note we provide a short proof of the associativity of Δ. This proof is not new. A slightly different version appears in Yousefnia [**4**]. However, the proof is not readily accessible to anyone unfamiliar with Persian.

Consider three sets A, B, and C. We define our universe to be $X = A \cup B \cup C$. For any subset U of X, define the characteristic function of U by

$$\chi_U(x) = \begin{cases} 1, & \text{if } x \in U; \\ 0, & \text{if } x \in X \setminus U. \end{cases}$$

Two subsets U and V of X are equal if and only if $\chi_U = \chi_V$. The following lemma is the key to our proof.

LEMMA. *For any two subsets U and V of X and for any $x \in X$,*

$$\begin{aligned} \chi_{U \Delta V}(x) &= (\chi_U(x) - \chi_V(x))^2 && (1) \\ &= \chi_U(x) + \chi_V(x) - 2\chi_U(x)\chi_V(x). && (2) \end{aligned}$$

Proof. Note that both sides of (1) are equal to 1 exactly when x belongs to one of U or V, but not to both. The identity (2) follows immediately from (1) and the fact that $\chi_S^2 = \chi_S$ for any set S. ■

PROPOSITION. *Let A, B, and C be three sets. Then*

$$(A \Delta B) \Delta C = A \Delta (B \Delta C).$$

Proof. From the Lemma we see that

$$\begin{aligned} \chi_{(A \Delta B) \Delta C} &= \chi_{A \Delta B} + \chi_C - 2\chi_{A \Delta B}\chi_C && (3) \\ &= (\chi_A + \chi_B - 2\chi_A\chi_B) + \chi_C - 2(\chi_A + \chi_B - 2\chi_A\chi_B)\chi_C \\ &= \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C. \end{aligned}$$

Since the last line in (3) is symmetric with respect to A, B, and C, we conclude that

$$\chi_{(A\Delta B)\Delta C} = \chi_{(B\Delta C)\Delta A}. \tag{4}$$

But Δ is commutative, and therefore (4) completes the proof of the Proposition. ■

By using modular arithmetic, we can make the above proof even shorter. Note that $U = V$ if and only if $\chi_U \equiv \chi_V \pmod 2$. Thus, the Lemma becomes

$$\chi_{U\Delta V} \equiv \chi_U + \chi_V \pmod 2.$$

The associativity of the symmetric difference now follows from the associativity of addition in modular arithmetic.

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