

## Taylor Series

The activities described here will help you become comfortable using the Taylor Series applet. In the first activity, we will become familiar with the features of the applet. In the second activity, we will use the applet to explore Taylor approximations in greater depth.

**Theorem 1** (Taylor’s Theorem). *Let  $f$  and its first  $n$  derivatives be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and let  $c \in [a, b]$ . Then for each  $x \in [a, b]$  with  $x \neq c$ , there exists a point  $\gamma$  between  $x$  and  $c$  such that*

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \frac{f^{(n+1)}(\gamma)}{(n + 1)!}(x - c)^{n+1}.$$

**Activity 1** (Getting comfortable with the applet). Begin by launching the Taylor Series applet in a web browser. Once it is loaded, you are ready to begin. When you first load the applet, you will see a lot of things on the screen. Let’s look at each of them in turn.

1. Begin by clicking the box next to “Show Error Function” to uncheck it. We will want to turn this back on later, but for now let’s leave it off. There should be two functions shown on the axes. The blue curve is  $f(x) = e^{-x^2}$  and is the function that we are planning to approximate via a Taylor expansion. The green curve is the Taylor polynomial.
2. On the  $x$ -axis is a point  $c$ . This is the point about which we are doing the approximation. Click on this point and slide it to the left and the right. Watch how the green curve changes as you do this.
3. Move  $c$  to 1 and leave it there for now. On the top left of the screen, in green text, you will see “Taylor Series of degree ...” which has a green slider underneath it. By manipulating this slider, you can adjust the degree of the polynomial approximation. Click the green point and slide it to the left until  $N = 1$ . You should see that the green curve has turned into a line, giving us a tangent line approximation to  $f(x)$  at  $x = c$ . Now slide the green point to the right and watch how the green curve changes to a higher degree polynomial to give a better fit for  $f(x)$ .
4. Now let’s click the “Show Error Function” button again to turn the Error function back on. A red dotted curve should appear on your screen. This curve represents the error, the difference between the function  $f(x)$  and the approximating polynomial. We keep track of the error on the interval  $[c - r, c + r]$ . You have already seen how to control  $c$ , and you can control  $r$  by adjusting the slider beneath  $c$ . Click on the slider and increase and decrease  $r$ . As you do this, watch how the error curve adjusts and watch how the maximum error changes in the top left corner.

5. Finally, you can change the function  $f(x)$  that you wish to explore by typing in the Input box at the bottom of the applet. Go to the Input box and try typing “ $f(x) = \sin(x)$ ” and then pressing Enter. You can always reset the applet to its default mode by clicking the icon in the top-right corner.

**Activity 2** (Exploring Taylor Series). Now that you are comfortable with all of the features of the applet, let’s explore Taylor approximations in more depth. Begin by resetting the applet to its default.

1. Set  $f(x) = e^x$  by typing “ $f(x) = e^x$ ” into the Input box. Set  $c = 0$  and  $r = 2$ . We are going to look at approximations of  $e^x$  on the interval  $[-2, 2]$  centered around  $c = 0$ . For each of the indicated degrees  $N$ , record the maximum error of the approximation on the interval  $[-2, 2]$  given by the degree  $N$  approximation:  $N = 1, 3, 5, 7, 9$ .
2. Taylor’s Theorem suggests that we can compute a theoretic maximum error of the degree  $N$  approximation by giving an upper bound for

$$\frac{f^{(N+1)}(x)}{(N+1)!}(r)^{N+1}$$

on the interval  $[-2, 2]$ . For each of the values of  $N$  in the previous problem, compute a theoretic upper bound on the error and then compare it to the actual maximum error.

3. Repeat items 1 and 2 with  $c = 5$  and  $r = 2$ . You won’t be able to see the approximation as well, but you can still get the maximum error from the applet.
4. Repeat items 1 and 2 using  $f(x) = \sin(x)$ .
5. Repeat items 1 and 2 using  $f(x) = \sqrt{x}$  centered at  $c = 4$ .
6. Write a paragraph setting out guidelines for predicting how well Taylor polynomials will perform as approximating functions for different types of functions. Specifically, what features of a function will cause a large error to appear (thus necessitating a high degree approximation to obtain any accuracy) and what features of a function will suggest the function can be approximated fairly accurately with a low degree polynomial?