Antoine's Necklace or How to Keep a Necklace From Falling Apart

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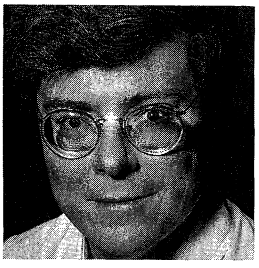
Beverly L. Brechner was born in New York City, where she was educated in her early years. She completed high school in Miami Beach, and received her Bachelor's and Masters' degrees from the University of Miami in Coral Gables, Florida. She received her Ph.D. in mathematics (topology) in 1964 from Louisiana State University in Baton Rouge and taught for several years at Louisiana State University in New Orleans. She then accepted a position at the University of Florida, where she is now Professor of Mathematics. During her tenure at Florida, she spent a sabbatical year at the University of Michigan, as well as a summer at the University of Texas, doing research and teaching. Her mathematical interests include geometric topology and continuum theory.



Born in Pennsylvania and educated in his early years in New Jersey and Florida, John C. Mayer received his baccalaureate degree in 1967 from Randolph-Macon College in Ashland, Virginia, and, after nine years teaching secondary school, doctorates in philosophy (logic) in 1980 and mathematics (topology) in 1982 from the University of Florida (where Beverly L. Brechner was his mathematics advisor). After a two-year postdoctoral fellowship at the University of Saskatchewan, he accepted a position at the University of Alabama at Birmingham, where he is at present. His mathematical interests include geometric topology, continuum theory, and, lately, dynamical systems.

Introduction. Imagine a string of beads, but without the string, forming a necklace that *cannot fall apart*. "Impossible!" you say. Perhaps, if you are being practical. But theoretically speaking, if each of the beads were just a point, and if there were uncountably many of them, it would indeed be possible. This is just one of the many very surprising (and, therefore, very interesting) results in geometric topology.

We present below a construction, due to Louis Antoine [1] and called Antoine's Necklace, that does precisely what is described above. We will also discuss some well-known applications and generalizations of Antoine's Necklace, including the fact that there are infinitely (in fact, uncountably) many *different* Antoine's Necklaces. All our examples will be subsets of Euclidean spaces. The three familiar Euclidean spaces are the line (E^1), the plane (E^2), and three-space (E^3).

Cantor sets. The standard *Cantor set* C is constructed by deleting the middle third of the unit interval $C_0 = I = [0,1]$, and then successively deleting the middle third of each resulting subinterval (Figure 1a).

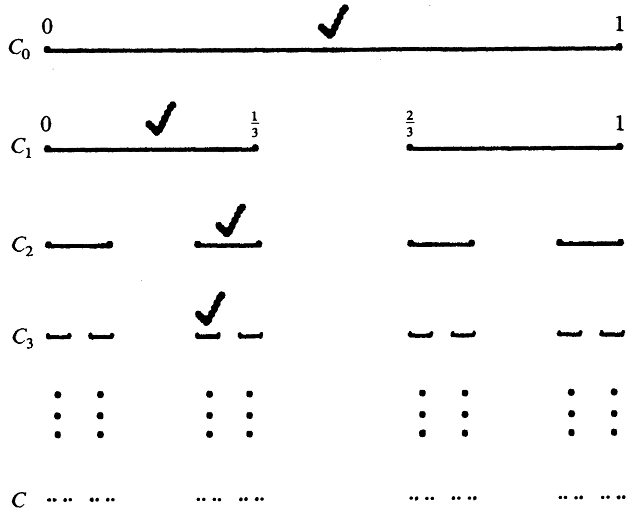


Figure 1a. The deleted middle-third Cantor set.

Figure 1b. A nonendpoint of the Cantor set: the intersection of checked subintervals.

Each C_n is the union of 2^n closed intervals of length 3^{-n}. Since the nested intersection of compact (closed and bounded) sets is nonempty, the Cantor set C may be (and is) defined as the intersection of these unions; that is, $C = \bigcap_{n=0}^{\infty} C_n$ is the set of points remaining after deleting the open middle thirds, *ad infinitum*. Since we have deleted open middle-third intervals, C contains all the endpoints of these deleted intervals. Since there are only countably many of these deleted intervals, there are only countably many of these endpoints. It is also well known that C consists of uncountably many points! An example of a point $p \in C$ that is not an endpoint of a deleted interval may be found by taking p to be the intersection of the subintervals of C_i in Figure 1b, where we alternately choose left and right subintervals of the previously chosen interval. This is necessary since a point $q \in C$ is an endpoint of some deleted interval if and only if it is a left (or right) endpoint of a *remaining* interval at some stage, and it always remains a left (or always remains a right) endpoint of a remaining interval at each subsequent stage. Thus, to find a point of C that is not an endpoint, we must alternately choose left and right subintervals infinitely often.

There are other subsets of Euclidean space that have the same properties as C. A subset F of a Euclidean space E is called a *Cantor set* if and only if F is

1. totally disconnected (the only connected set containing a given point of F is that point itself)
2. compact (closed and bounded)
3. perfect (every point of F is a limit point of F; that is, $x \in F$ if and only if $x = \lim_{n \to \infty} x_n$ for $\{x_n\} \subseteq F - \{x\}$).

It can be shown [2, p. 175] that a subset F of E has all three properties if and only if it is homeomorphic (see below) to the standard Cantor set C.

Embeddings and equivalence. For us, all spaces will be subsets of the real line E^1, the Euclidean plane E^2, or Euclidean three-space E^3. A *homeomorphism* of a space X *onto* a space Y is a one-to-one continuous function h: $X \to Y$ whose inverse h^{-1}: $Y \to X$ is also continuous. If such a homeomorphism exists, then X and Y are said to be *homeomorphic*. For example, a rubber band, whose natural shape is oval, can be stretched into a triangle, or a square, or even quite irregular shapes, and each such deformation is a homeomorphism. One can even cut the rubber band and glue the ends back together again after knotting it, and the result will still be homeomorphic to the original. For more concrete examples, one can readily verify that $h(x) = x/(1+|x|)$ is a homeomorphism of $X = E^1$ onto $Y = (-1,1)$, whereas the one-to-one continuous function $g(t) = (\cos 2\pi t, \sin 2\pi t)$ of $X = [0,1)$ onto $Y = \{(x, y)\colon x^2 + y^2 = 1\}$ is not a homeomorphism because g^{-1} fails to be continuous at $(1,0)$. Note, however, that g is a homeomorphism of the open interval $(0,1)$ onto the punctured circle $Y - \{(1,0)\}$.

If h: $X \to Y$ is a homeomorphism of X onto Y, and $Y \subseteq E$, we say that h is an *embedding* of X in E. For example, there is a homeomorphism from the unit circle onto the square having corners $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$ in the plane. This homeomorphism can then be considered as an embedding of the unit circle into E^2. We shall also refer to $Y = h(X)$, the image of X under h, as an *embedding* of X in E. Thus, embedding may refer to a function or to its image, depending on context. The last remark of the preceding section asserts that every Cantor set in E^2 or E^3 is an embedding of the standard Cantor set C.

In Figure 2a below, A is a circle with a sticker glued to its exterior, and B is a circle with a sticker glued to its interior. The figures A and B are homeomorphic. Any homeomorphism of A onto B must carry the circle to the circle and the sticker to the sticker. However, for a homeomorphism of the plane onto itself, more is required. If one were to draw A on a rubber sheet, one could not deform it into B by any amount of stretching and pulling. That is, there is no homeomorphism of E^2 (or, analogously, the rubber sheet) onto itself taking A onto B. This "rubber sheet geometry" captures intuitively the essence of the Jordan-Schoenflies Theorem: *any homeomorphism of the plane onto itself that takes the circle in A to the circle in B must also take the interior of the circle in A to the interior of the circle in B, and the exterior of the circle in A to the exterior of the circle in B.* Therefore, such a homeomorphism cannot take the sticker to the sticker.

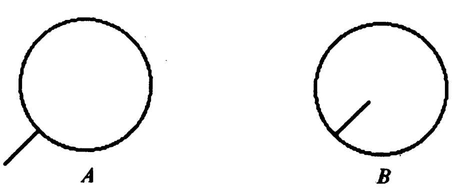


Figure 2a. Inequivalent embeddings.

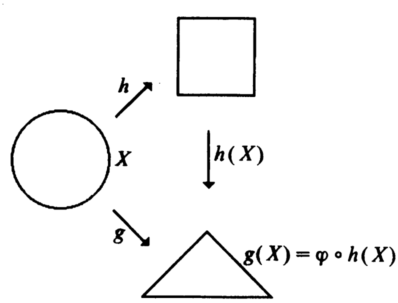


Figure 2b. Equivalent embeddings.

In view of the preceding remarks, we shall say that two subsets A and B of a Euclidean space E are *equivalent* if and only if there is a homeomorphism h: $E \to E$ of the **space E onto itself**, that takes A onto B. (This also establishes that A and $B = h(A)$ are homeomorphic.)

Similarly (Figure 2b), two embeddings $g, h\colon X \to E$ are called *equivalent* (and the sets $g(X)$ and $h(X)$ are said to be *equivalently embedded*) if and only if there is an onto homeomorphism $\varphi\colon E \to E$ such that $\varphi(g(X)) = h(X)$. (Note, therefore, that $g(X)$ and $h(X)$ are equivalent subsets of E.) For example, a square and a triangle are equivalently embedded images in E^2 of the unit circle. (Imagining the plane as a rubber sheet, one can stretch the plane so that the triangle becomes the square.)

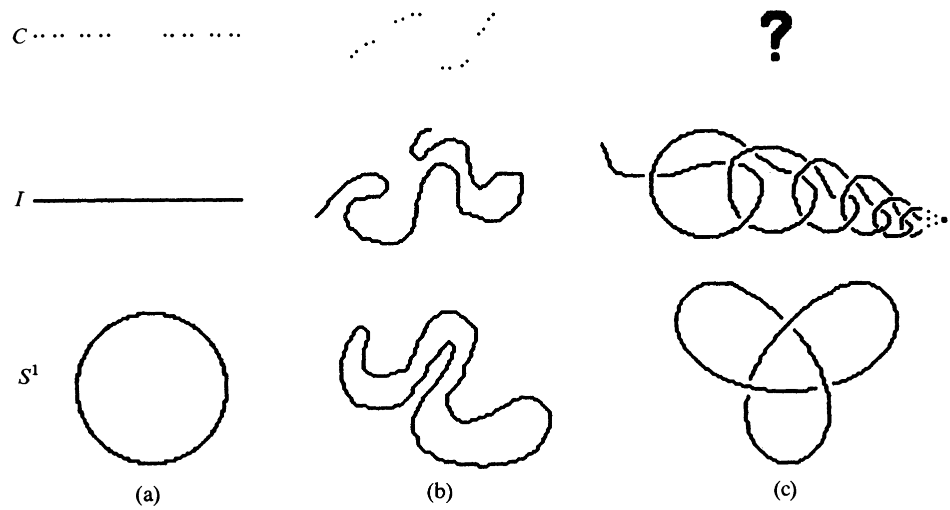
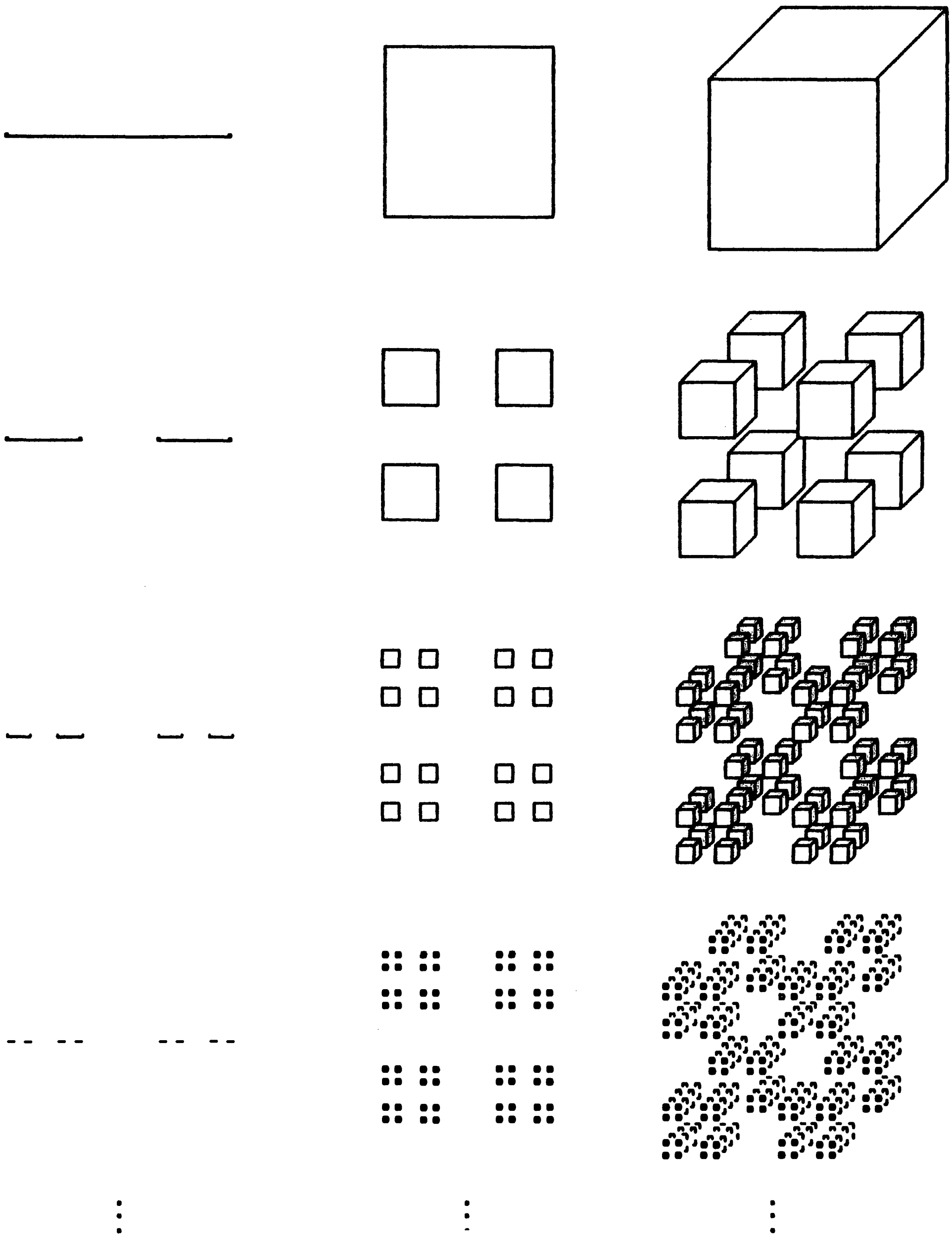


Figure 3. Embeddings of the Cantor set, interval, and circle.

In Figure 3(a), we illustrate the standard embeddings in E^2 of the Cantor set C, the unit interval $I = [0,1]$, and the circle S^1 (the graph of $x^2 + y^2 = 1$). Since E^2 is a subspace of E^3 (the xy-plane in xyz-space, if you like), we can also view these illustrations as embeddings in E^3. Figure 3(b) shows different-looking, equivalent embeddings in E^2 (or E^3, if you prefer) of C, I, and S^1, respectively.

The embeddings in 3(b) are called *tame* embeddings. That the figures in 3(b) are equivalent to those in 3(a) is a consequence of Theorem 1 below. For the unit interval and circle, Figure 3(c) shows embeddings in E^3 that are *not equivalent* to the standard embeddings. The proof of this fact requires some knowledge of algebraic topology—specifically homotopy theory and the fundamental group [3], [4]. Intuitively, it is the "crossovers" that do not allow equivalent embeddings in E^3. The embedding of the interval I in Figure 3(c) is called a *Fox-Artin arc* [3], and is an example of a *wild* embedding of I into E^3. The embedding of the circle S^1 in Figure 3(c) is called a *trefoil knot*, and is an example of a *knotted* embedding of S^1 into E^3.

Figure 4 illustrates three constructions of Cantor sets. Figure 4(a) depicts the construction of a Cantor set as the intersection of intervals in the real line. Figure 4(b) shows the construction of a Cantor set as the intersection of a sequence of sets, each of which is a collection of square-plus-interiors in the plane E^2. Figure 4(c) shows the construction of a Cantor set as the intersection of cubes in E^3. Note that the size or *diameter* of each individual square or cube goes to zero as the number of stages increases to infinity. These sets can be recognized as Cantor sets because each



(a) (b) (c)

Figure 4.

is compact, perfect, and totally disconnected. Moreover, as we shall show, the Cantor set in Figure 4(b) is equivalent in E^2 to the Cantor set in Figure 4(a). Since E^1 and E^2 are subsets of E^3, each of these Cantor sets can be regarded as a subset of E^3. In fact, the three Cantor sets in Figure 4 are equivalent subsets of E^3, since there is a standard method for constructing homeomorphisms of E^3 onto itself carrying any one of these Cantor sets onto any one of the others. We do not know a reference, but we indicate a proof in Figure 5.

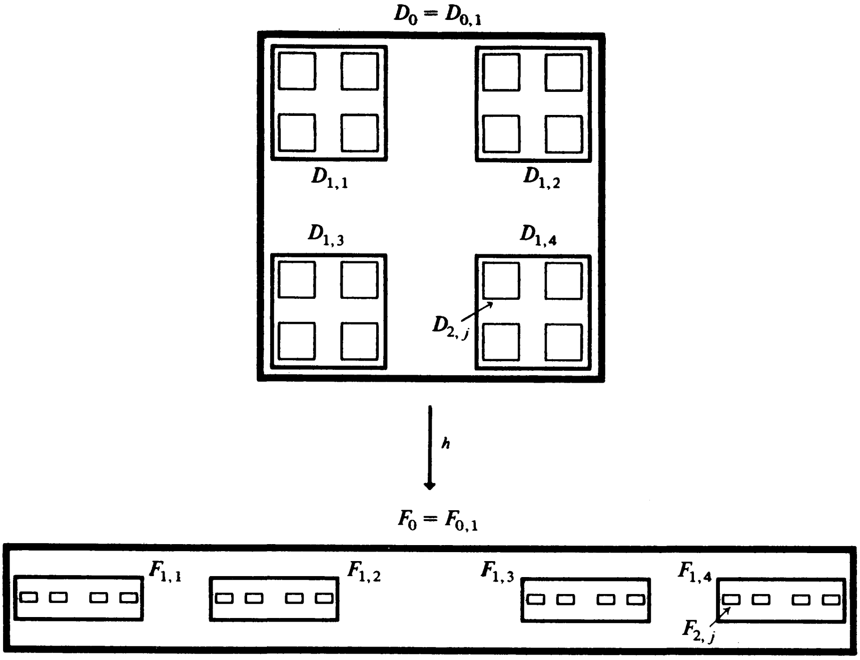


Figure 5.

We describe briefly how to construct a homeomorphism of E^2 onto E^2 which carries the Cantor set in Figure 4(a) onto the Cantor set in Figure 4(b). Note first (Figure 5) that the Cantor set in Figure 4(b) may be described as the intersection of a sequence of sets $\{D_i\colon i = 0, 1, 2, \ldots\}$, where

$$D_i = D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,4^i}$$

is the union of 4^i (square-plus-interior)s. Since $\cap_{i=0}^{\infty} C_i = \cap_{i=0}^{\infty} C_{2i}$, the Cantor set in Figure 4(a) may be described as the intersection of a sequence of sets $\{F_i\colon i = 0, 2, \ldots\}$, where

$$F_i = F_{i,1} \cup F_{i,2} \cup \cdots \cup F_{i,4^i}$$

is the union of 4^i (rectangle-plus-interior)s. To see this, let F_0 be C_0 thickened slightly to form a rectangle, and let the thickness of the intervals in C_{i+1} be less than the thickness of the intervals in C_i. Then for each stage C_{2i} of the intervals, we have the stage F_i consisting of 4^i rectangles. Each rectangle of F_{i-1} contains 4 rectangles of F_i. Note that the intersection of the sets of intervals is the same as the intersection of the sets of rectangles, provided that the thickness of the intervals (heights of the rectangles) decreases to zero. Thus, the standard Cantor set C may be constructed as the intersection of sets of rectangles in the plane.

Let D_i and F_i be as labelled in Figure 5. Then the homeomorphism $h\colon E^2 \to E^2$ that carries the Cantor set in Figure 4(b) onto the Cantor set in Figure 4(a) is obtained as the limit of a sequence of homeomorphisms $\{h_i\colon E^2 \to E^2 (i = 0, 1, 2, \ldots)\}$, where h_0 carries the square-plus-interior D_0 to the rectangle-plus-interior F_0, and in general where h_{i+1} is a modification of h_i on the interiors of the 4^i rectangles of $h_i(D_i)$, essentially making the 4^{i+1} rectangles of $h_i(F_{i+1})$ "horizontal," within the preceding rectangles.

The equivalence of the Cantor sets in Figures 4(a) and 4(b) is a special case of Part (1) of the following Theorem (see [1], [2]):

Theorem 1. (1) *Every two embeddings in* E^2 *of a Cantor set are equivalent;*

(2) *Every two embeddings in* E^2 *of an* arc (*that is, a homeomorphic image of* $[0,1]$) *are equivalent;*

(3) *Every two embeddings in* E^2 *of the unit circle* S^1 *are equivalent.*

One might wonder whether E^2 can be replaced by E^3 in the theorem. The answer is "no" for each of the three statements of the theorem. Antoine's Theorem (below) shows that there exist Cantor sets that are inequivalently embedded in E^3. There are also inequivalently embedded arcs and inequivalently embedded circles in E^3. Classical proofs use algebraic topology. Our Theorem 3 shows how to obtain inequivalently embedded arcs and circles in E^3 as a consequence of the existence of inequivalently embedded Cantor sets.

Antoine's Necklace. Antoine's Necklace is a Cantor set A in E^3 that is *not* equivalent to the standard Cantor set C. This necklace is obtained as the intersection of an infinite sequence of collections of solid tori (doughnuts), whose diameters decrease to zero.

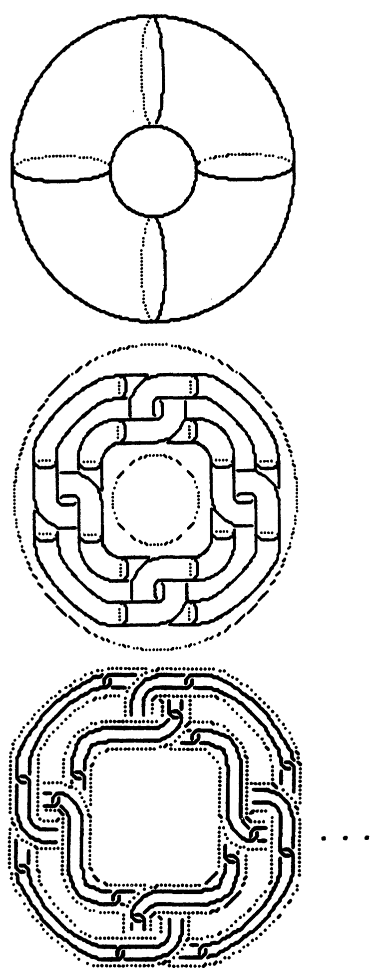


Figure 6. Stage-by-stage construction of Antoine's Necklace.

Figure 6 illustrates how the tori of each stage sit inside, or refine, the tori of the preceding stage. We begin with a single torus $T_0 = T_{0,1}$ in the 0th stage. At each subsequent stage, four smaller tori inside *each* torus of the preceding stage are linked together to form a circular chain around the hole of the torus that the circular chain refines.

At the ith stage, we have a collection

$$T_i = T_{i,1} \cup T_{i,2} \cup \cdots \cup T_{i,4^i}$$

of 4^i tori. Note that the diameters of the $T_{i,j}$'s ($j = 1, 2, \ldots, 4^i$) decrease to zero as i increases to infinity. (While it is clear that we can make the diameters of the tori in the refining collections tend to zero, it is *not* clear that this can be done with the same number of tori refining each torus. This, however, can be done, and a proof may be found in [1].) *Antoine's Necklace A* is defined as

$$A = T_0 \cap T_1 \cap T_2 \cap T_3 \cap \cdots,$$

the infinite intersection of these unions of tori.

Since the construction process is continued ad infinitum, the Cantor set lies in the interior of each union T_i of doughnuts—in fact, every complete cross-sectional chunk or "bite" of any doughnut will contain certain points of A. Figure 7 details the first four stages of construction.

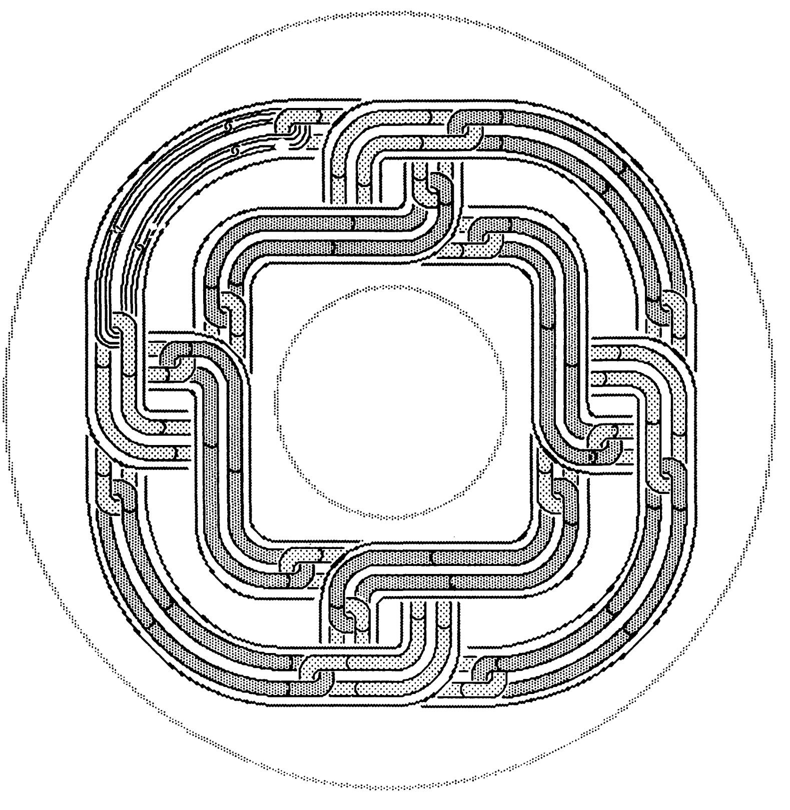


Figure 7. First four stages in the construction of Antoine's Necklace.

By analogy with C, where intervals are used to construct C but no interval is contained in C (the set C consists only of the points of intersection of these intervals), the tori are used to construct Antoine's Necklace, but *no torus is actually contained in* Antoine's Necklace. Only the "beads," the points that are intersections of (infinitely many) solid tori, are left. Antoine's Necklace is totally disconnected, compact, and perfect. It is totally disconnected because for any two different points, there is some stage of the construction such that the two points will lie in different tori (since the diameter of individual tori goes to zero as i goes to infinity). Antoine's Necklace is compact because it is the intersection of the closed and bounded collections of tori. It is perfect because every torus contains at least two tori at the following stage. This allows us to choose, for any point $x \in A$, a sequence from $A - \{x\}$ converging to x, so that each point of A is a limit point of A. Thus, by the characterization of the Cantor set given earlier, A is a Cantor set homeomorphic to C.

Antone's Necklace A looks like the beads of a necklace; there is no string (since A is totally disconnected), but it cannot fall apart!

To explain what we mean by "cannot fall apart," we use the following terminology. A *sphere* in E^3 is a homeomorphic image of the unit sphere $\{(x, y, z): x^2 + y^2 + z^2 = 1\}$ in 3-space. We say that a set A *cannot be separated by a sphere* if for every sphere S that contains some point of A inside S and some point of A outside S, then some point of A must lie *on* S.

Antoine's Lemma. *Antoine's Necklace cannot be separated by a sphere.*

The details of the proof of this Lemma are beyond the scope of this paper. However, the heart of the proof involves showing that any sphere which contains a point of A in its interior must intersect a torus of every stage. For the interested reader, a geometric proof may be found in Antoine's paper [1] (in French), whereas a proof using homotopy theory may be found in [4]. Note that the standard Cantor set C in E^3 *can* be so separated, as illustrated in Figure 8.

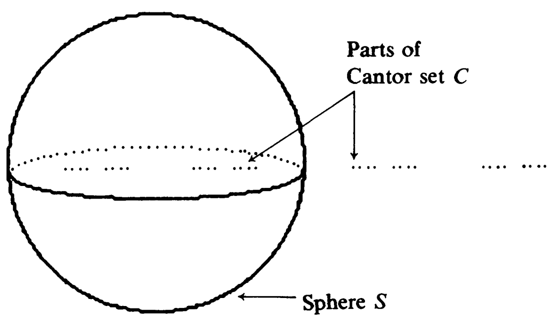


Figure 8. A sphere separating the Cantor set C in E^3.

Although C and A are homeomorphic, they are not equivalently embedded in E^3. To see why, assume that C and A are equivalently embedded in E^3. Then there would be a homeomorphism h of E^3 onto itself such that $h(C) = A$. Let S be a sphere separating C, as illustrated in Figure 8. Then $h(S)$ is also a sphere (by definition). To see that $h(S)$ separates A, let $C = C_I \cup C_E$, where C_I and C_E denote the parts of C interior and exterior to S, respectively. Then, since a homeomorphism maps open sets to open sets (and also closed sets to closed sets), $h(C) = h(C_I) \cup h(C_E)$ is a disjoint union of two open subsets of $h(C)$ such that $h(C_I)$ is interior to $h(S)$ and $h(C_E)$ is exterior to $h(S)$. Thus, $A = h(C)$ is separated in E^3 by the sphere $h(S)$. Since this conclusion contradicts Antoine's Lemma, we have the following:

Antoine's Theorem. *The standard Cantor set C and Antoine's Necklace A are not equivalently embedded in* E^3.

Uncountably many Antoine's Necklaces. Now that we have two different (that is, inequivalent) embeddings in E^3 of the Cantor set—the standard one C and Antoine's Necklace A—one might wonder whether there are others. Instead of using four tori within each torus to construct Antoine's Necklace, could we use five tori? Would the necklace obtained be different—that is, would the resulting embedding in E^3 of the Cantor set be inequivalent to the previous two embeddings?

The answer is affirmative. In fact, the following theorem [5], which we state without proof, is the basis for creating uncountably many inequivalent Antoine's Necklaces (embeddings of the Cantor set) in E^3.

Sher's Theorem. *Two Antoine's Necklaces*

$$A = \bigcap_{i=0}^{\infty} T_i \quad \textit{and} \quad B = \bigcap_{i=0}^{\infty} R_i$$

are equivalently embedded in E^3 *if and only if at corresponding stages of their constructions, they always have the same number of refining tori and they link in the same way. That is, there is a homeomorphism h of* E^3 *onto itself such that* $h(T_i) = R_i$ *for all* $i = 0, 1, 2, \ldots$.

In order to obtain an uncountable collection $\{A_\alpha\colon\ \alpha \in \mathscr{A}\}$ of inequivalent Antoine's Necklaces, it suffices to produce uncountably many sequences $\{S_\alpha\colon\ \alpha \in \mathscr{A}\}$ of positive integers that differ pairwise in at least one entry. For any such sequence S_α, the nth entry will denote the number of tori that will refine each torus of the preceding $(n-1)$th stage in the construction of A_α. The tree diagram in Figure 9 shows how we can get infinitely (in fact, uncountably) many distinct sequences of integers, using only the integers 4 and 5. Each path through the tree represents such a sequence, and any two such sequences clearly differ in at least one entry. Figure 10 illustrates the first four stages in the construction of the necklace that results from the sequence 1-4-5-4.... Compare this to Figure 7.

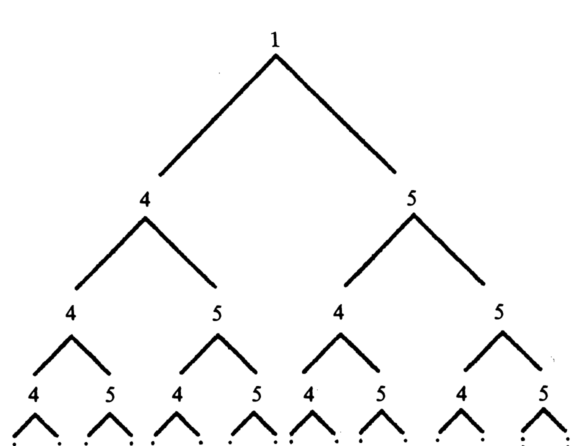


Figure 9. An infinite tree of positive integers.

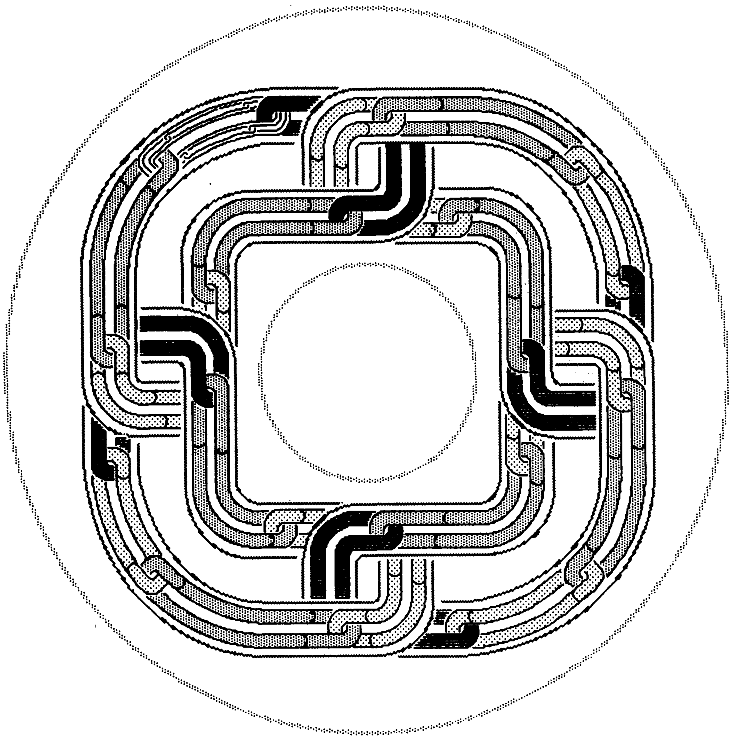


Figure 10. First four stages in the construction of a necklace, using 1-4-5-4 refining tori.

Our goal. In the preceding sections, we have shown that there are uncountably many inequivalent Antoine's Necklaces in E^3. The goal of the remaining sections is to extend this result to sets other than the Cantor set. Specifically, we will sketch a proof of the following:

Theorem 2. *Let X be any nondegenerate planar continuum (that is, a closed, bounded, and connected subset of the plane). Then there exist uncountably many inequivalent embeddings of X into E^3.*

Outline of proof. There are three main steps:

(1) Show that there are uncountably many inequivalent Antoine's Necklaces $\{A_\alpha: \alpha \in \mathscr{A}\}$ in E^3.
(2) Construct uncountably many inequivalent disks $\{D_\alpha: \alpha \in \mathscr{A}\}$ in E^3 (Antoine's Horned Disks), where $D_\alpha \supseteq A_\alpha$ for each $\alpha \in \mathscr{A}$.
(3) Embed X as X_α in E_3 in such a way that

$$A_\alpha \subseteq X_\alpha \subseteq D_\alpha$$

and the points of A_α are the only points of D_α at which D_α is not *locally planar* (defined in the next section).

Based on (1)–(3), we shall conclude that $\{X_\alpha: \alpha \in \mathscr{A}\}$ is the desired collection of inequivalent embeddings of the planar continuum X in E^3.

Step (1) has already been accomplished. The next section accomplishes steps (2) and (3).

Construction of Antoine's Horned Disks: Reach out and touch it. Consider A_α, already embedded in E^3 as the intersection of stages consisting of collections of tori. Start with a *unit disk* $D = \{(x, y): x^2 + y^2 \leq 1\} \subseteq E^2 \subseteq E^3$. Think of this unit disk as made of rubber, and stage-by-stage pull out "feelers," making a "feeler disk" reaching out to all the tori at that stage. Figures 11 and 12 illustrate some of the first stages in the development of D_α. (See pages 76–79 of [4] for other illustrations and related theorems.)

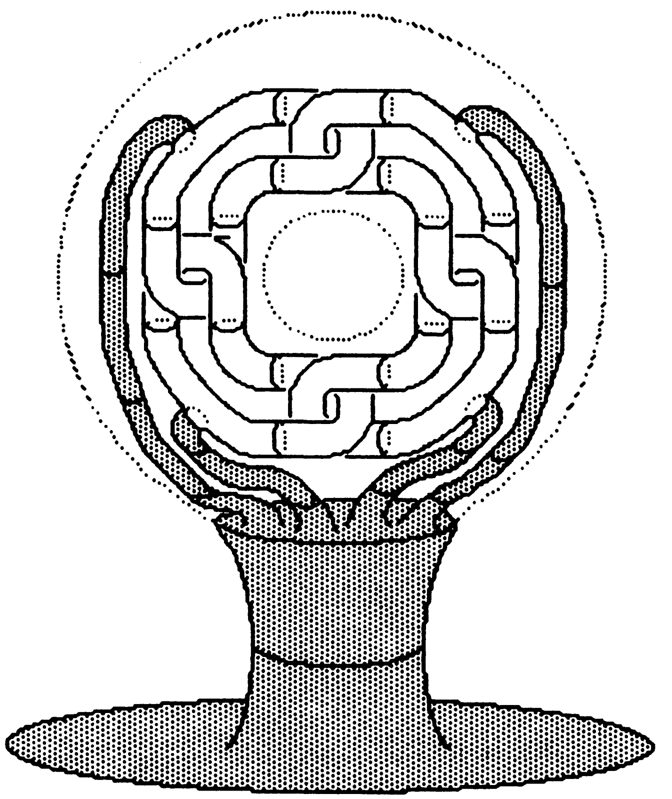


Figure 11. Hollow "feelers" of disk reaching out to touch tori. First two stages in the construction of the limit disk D_α.

Continue stage-by-stage until the multiply-branching feelers (or horns) reach all the points of A_α in the limit. The resulting set D_α, which contains A_α, is called an *Antoine's Horned Disk*. This limit *disk* D_α is, in fact, a homeomorphic copy of the unit disk D (since it is, in essence, a continuous deformation of D). Therefore, D_α is an embedding of the unit disk into E^3. However, D_α and D are not equivalently embedded in E^3. The difference is that D is locally planar at *each* of its points, while Antoine's Horned Disk D_α is locally planar *only* at those of its points *not* in A_α. (A disk Γ is *locally planar* at a point $\gamma \in \Gamma$ when a neighborhood in Γ of γ can be mapped into the *xy*-plane by a homeomorphism of E^3 onto itself.) That D_α *is* locally planar at any point $p \in D_\alpha - A_\alpha$ essentially follows from the fact that, in the construction of D_α, one reaches the point p after only a finite number of stages. Thus, a neighborhood of p can be "pushed back" into D, which is planar, after a finite number of reverse deformations. One needs to use the fundamental group, and homotopy theory, to actually prove that D_α *is not* locally planar at the points of A_α.

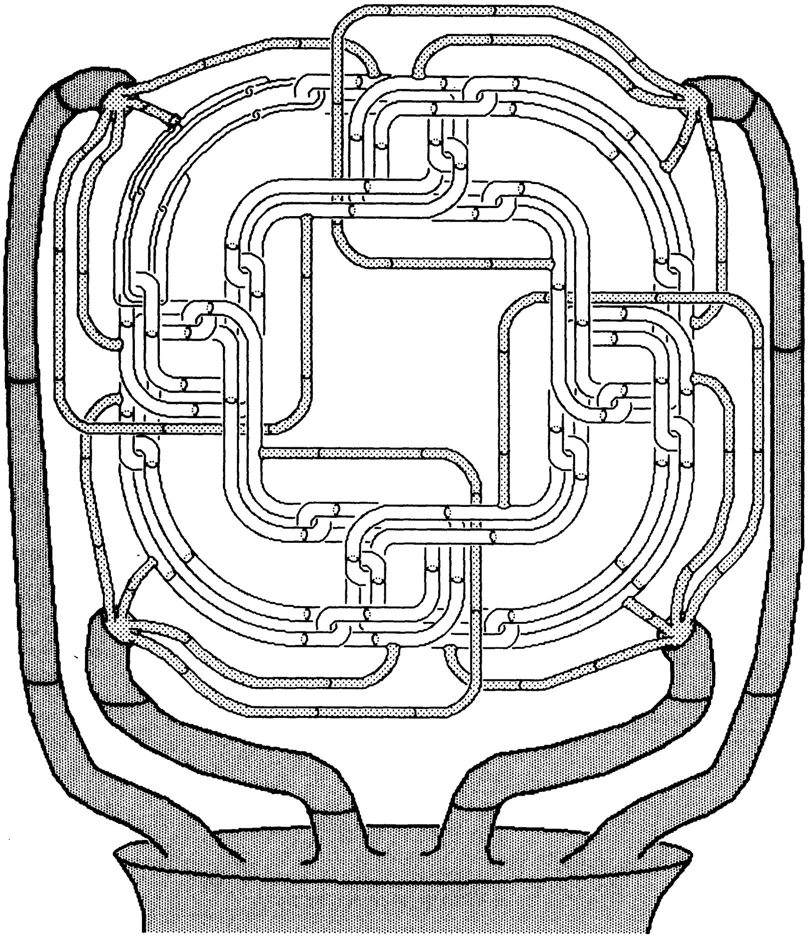


Figure 12. First four stages of "feelers" shown in the development of D_α.

Since the property of a set being, or not being, locally planar at a point is preserved by a homeomorphism of E^3 onto E^3, we see that if A_α and A_β are inequivalent Antoine's Necklaces, then D_α and D_β are inequivalent Antoine's Horned Disks. (If D_α and D_β were equivalent, then $D_\beta = h(D_\alpha)$ for some homeomorphism h of E^3 onto E^3. Since $A_\alpha = \{$points where D_α is not locally planar$\}$, it follows that $h(A_\alpha) = \{$points where D_β is not locally planar$\} = A_\beta$, and therefore that A_α is equivalent to A_β.) Thus, there are uncountably many inequivalent Antoine's Horned Disks in E^3. This completes our verification of step (2) in Theorem 2.

Theorem 3. *Let X be a nondegenerate planar continuum. Then there are uncountably many inequivalent embeddings $\{X_\alpha\colon \alpha \in \mathscr{A}\}$ of X into E^3 such that*

$$A_\alpha \subseteq X_\alpha \subseteq D_\alpha \subseteq E^3$$

and D_α is locally planar at every point not in A_α.

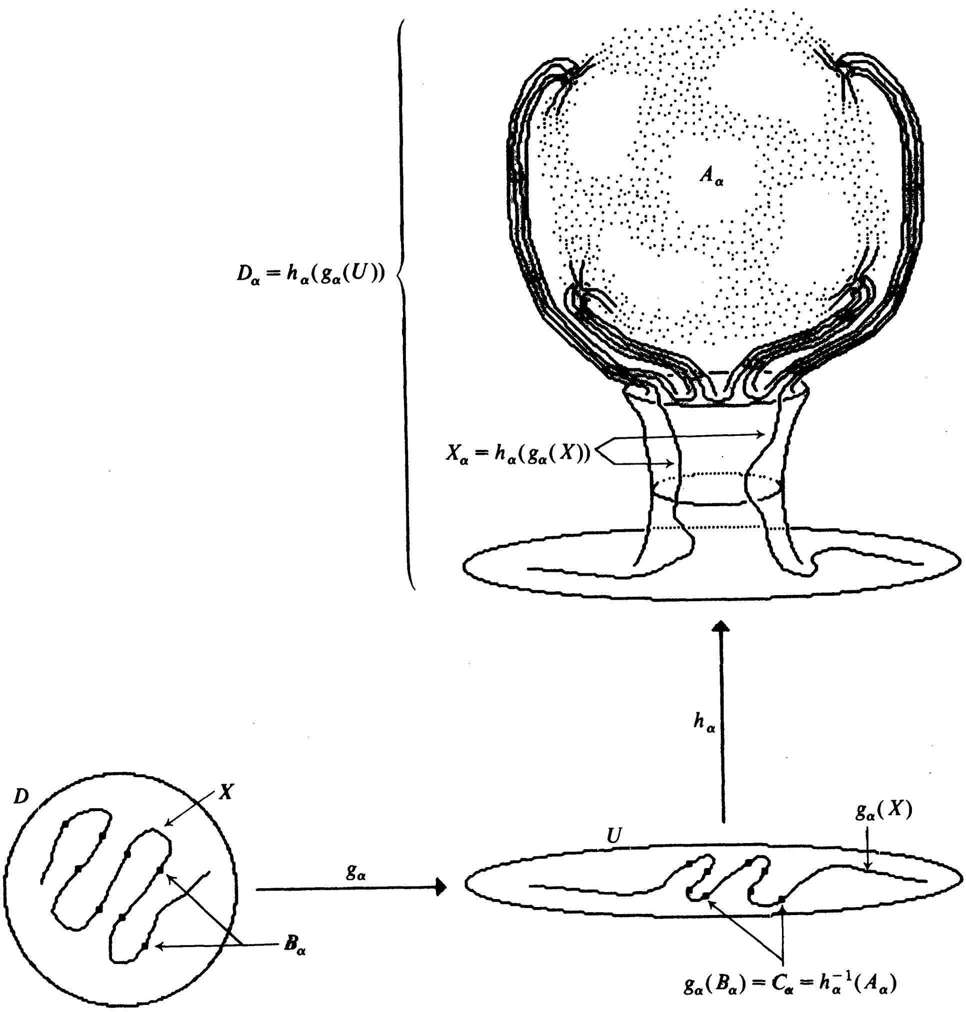


Figure 13. Embedding X in E^3 as $X_\alpha \supseteq A_\alpha$.

Proof. The proof is illustrated by Figure 13, and follows the steps outlined below:

(1) Let $D = \{(x, y)\colon x^2 + y^2 \leq r^2\}$ be a disk in E^2 such that X is contained in the interior of D, and let $U = \{(x, y)\colon x^2 + y^2 \leq 1\}$ be the unit disk in E^2.

(2) For each $\alpha \in \mathscr{A}$, take U as the base of Antoine's Horned Disk D_α before D_α is formed, and let $h_\alpha\colon U \to D_\alpha \subseteq E^3$ be the embedding of U in E^3 determined by the construction of D_α.

(3) Since $h_\alpha^{-1}\colon D_\alpha \to U$ is an onto homeomorphism, and since homeomorphisms preserve the properties of being totally disconnected, compact, and perfect (the three characterizing properties of a Cantor set), it follows that $C_\alpha = h_\alpha^{-1}(A_\alpha)$ is a Cantor set in U that maps onto A_α under h_α.

(4) Let B_α be any Cantor set in X. Every nondegenerate continuum in E^2 has a Cantor set. (In fact, a Cantor set can be constructed by a process similar to the construction of the standard Cantor set C.)

(5) In Theorem 1, we noted that any two Cantor sets C_1, C_2 in E^2 are equivalent. Moreover, we know that any two disks D_1 and D_2 are homeomorphic. It can be further proved that if the interior of D_1 contains C_1 and the interior of D_2 contains C_2, there is a homeomorphism of D_1 onto D_2 which takes C_1 onto C_2. A complete proof of this requires the Schoenflies Theorem and its corollary (Exercise 5.E.3, page 157 of [2]). Given this information, suppose we let g_α: $D \to U$ be a homeomorphism of D onto U such that $g_\alpha(B_\alpha) = C_\alpha$. Then

$$U \supseteq g_\alpha(X) \supseteq g_\alpha(B_\alpha) = h_\alpha^{-1}(A_\alpha),$$

and so

$$D_\alpha \supseteq h_\alpha(g_\alpha(X)) \supseteq A_\alpha.$$

(6) Now let $X_\alpha = h_\alpha(g_\alpha(X))$. Then the only points of X_α at which D_α is not locally planar are in A_α. Thus, it follows that $\{X_\alpha\colon \alpha \in \mathscr{A}\}$ is an uncountable collection of inequivalent embeddings of X in E^3. For suppose that h is a homeomorphism of E^3 onto itself taking X_α onto X_β. Then h must map A_α (the nonlocally planar points of X_α) onto A_β (the nonlocally planar points of X_β), and this contradicts the fact that A_α and A_β are not equivalently embedded in E^3.

Remark. Theorems 2 and 3 remain true, if "continuum" is replaced by "compactum (closed and bounded subset of the plane) that contains a Cantor set." The proofs are identical to the above proofs.

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I believe that mathematical reality lies outside of us, and that our function is to discover, or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply notes on our observations.

G.H. Hardy