The Mercedes Knot Problem

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1. INTRODUCTION. Have you ever coiled a long extension cord in order to store it (e.g., after you have finished vacuum-cleaning or mowing the lawn)? After you unwind it, you may notice that it is twisted many times (see Figure 1). There is a mathematical problem that is closely related to this phenomenon. It can challenge even a non-mathematician. Imagine a ball in the middle of a room, connected with three elastic bands b, m, and w to the walls. The elastic bands, whose ends are fixed, can be moved in any way but cannot be torn. The question is whether it is possible to move them in such a way that the elastic band m is wrapped several times around the elastic band w without moving the ball (see Figure 2). This is a geometrical problem, somewhere between topology and knot theory, however our solution does not really use any topology and is based on simple combinatorics.

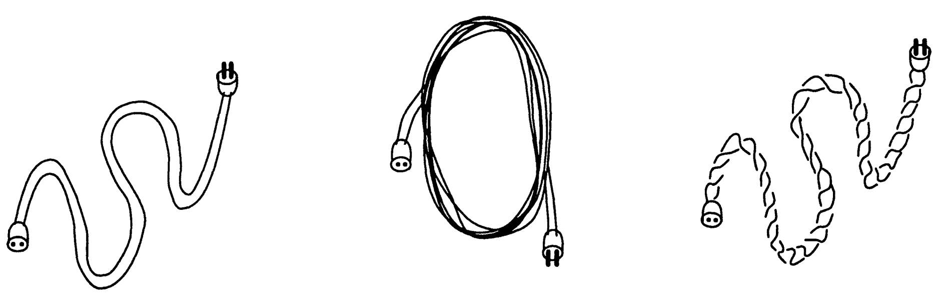


Figure 1. Why do the extension cords usually get twisted?

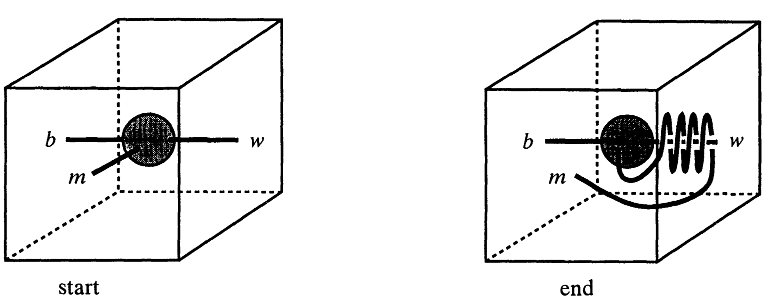


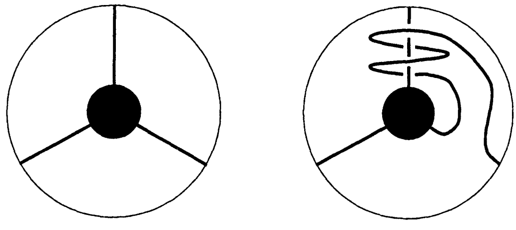
Figure 2

We first give a partial solution, then we translate the problem into mathematical language and give some historical background. In Section 4 we try to understand the gist of the problem by studying a few simpler situations from the knot theory point of view. After that we are able to give a complete solution. Along the way some interesting connections arise and at the end we mention some applications. Problem solvers will find a few interesting puzzles. We keep in mind Hilbert's advice: "*The art of doing mathematics consists in finding that special case which contains all the germs of generality.*"

2. PARTIAL SOLUTION. In a way this problem is mysterious, since from the very beginning our intuition may lead us in a wrong direction. Let us start with a few naive approaches. If we allowed the ball to rotate around its center, then the answer would trivially be 'yes': we could wrap the elastic band m around w as many times as we wanted (just by rotating the ball around the line determined by the ends of w), and the problem would not be interesting. We could reach a similar conclusion if we assumed for a moment that only the bands w and m were in the room. (Why?) But we cannot rotate the ball and there are three bands in the room, so maybe the answer is 'no'.

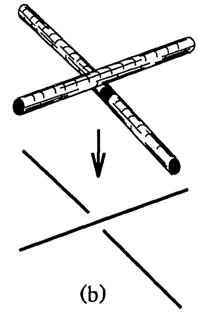
On the other hand, if we believe that the answer is 'yes', we may assume that this can be obtained even when the elastic bands b and w are rigid (do not move at all). This belief might be based on the fact that b and w are at the same place in both positions of Figure 2, or that the rotation of the ball would not have moved b and w (but it would twist them). So if the bands b and w are rigid, the space where m can be moved "looks like" a filled torus, which is what mathematicians call a donut (in topological language we would say "is homeomorphic to" $S^1 \times B^2$). But then there is no way to move the band m in a filled torus without moving its ends in order to change the number of times the band m is wrapped around the hole of the filled torus. A topologist would say that the starting and the ending position of m correspond respectively to the identity and a non-identity element of the fundamental group of the filled torus; see Neuwirth [**Neu**]. We conclude that either the movement of the elastic bands b and w is crucial, or the answer is 'no'.

Surprisingly, the answer (yes or no) depends on how many times we want to wrap m around w. Let us first start exploring the question of existence. By building a real model we find out unexpectedly quickly how to wrap the band m twice around the band w; see Figure 3(a), which also explains the title of this paper. As usual in knot theory we prefer to draw two-dimensional figures. They are basically projections (Figure 3(b) shows what happens at crossings) that contain all the necessary information to build the three dimensional objects they describe. Furthermore, the situation remains unchanged if we consider a two-dimensional sphere instead of a cube. An enthusiastic reader is invited to play with her/his own model before verifying our solution in Figure 4. We promise to the reader a better understanding of our solution by the end of this paper.



start end

(a)



(b)

Figure 3

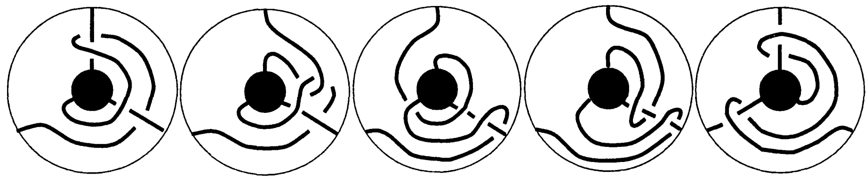


Figure 4. These are the intermediate steps between the above 'start' and 'end' position.

It is now obvious that, by moving the coils to the larger sphere and then repeating the procedure from Figure 4, we can wrap the band m around the band w any even number of times. The problem is much harder if we want to wrap the band m around the band w an odd number of times.

3. MODELING AND HISTORY OF THE PROBLEM. Let us translate the problem into mathematical language. As we are describing a geometrical problem, we will need some terminology from topology, however our solution requires very little topology. So let us follow topologists who prefer to speak about a motion of a space, called an *ambient isotopy*, instead of a movement of objects in it. To understand this better, imagine that the room is filled with honey (or any sticky substance). Then a movement of elastic bands in the room would cause honey to move. On the other hand, a motion of honey could move the elastic bands. So we usually ask for the existence of an ambient isotopy that moves objects from one position to another.

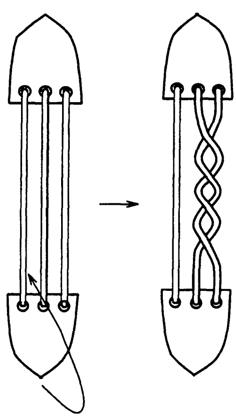
Let S_1 and S_2 be two-dimensional spheres with centers at the origin of $\mathbb{R}^3$ and radii one and two, respectively. Let H be the space between S_1 and S_2 (i.e., the hollow ball $S^2 \times [1,2]$), and let b, m, w be the line segments in which H intersects the positive coordinate axes x, y, z, respectively. In a conversation Jože Vrabec asked John Milnor the following question:

Question 1. *Is there an ambient isotopy of the hollow ball H fixing the spheres S_1 and S_2 that wraps m twice around w (see Figure 3(a))?*

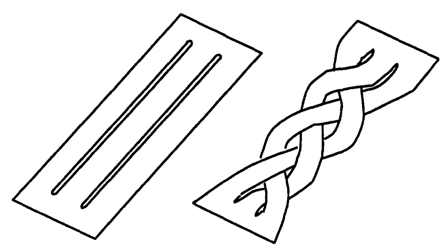
We have already seen a graphical proof at the end of the previous section. In his Topology I (1985/86) lectures, Professor Vrabec demonstrated John Milnor's topological solution; he used the nontrivial element of the fundamental group of $SO(3)$ ($\cong RP^3$) to build the desired ambient isotopy, see Appendix or [**Br**, pp. 164–167]. Intriguingly, neither our graphical solution nor his solution can be easily used to answer the remaining part of our problem:

Question 2. *Is there an ambient isotopy of the hollow ball H fixing the spheres S_1 and S_2 that wraps m once around w?*

It turns out that both questions have a long history. E. D. Bolker's paper [**Bo**], which appeared over 20 years ago in the Monthly, gives another topological solution of Question 1. Physicists were, however, the first scientists concerned with Question 1. They used its solution in order to better understand the quantum nature of the electron. Swiss physicist W. Pauli used the calculus of spinors to model electrons. His model implies that the electron has to spin for 4π radians in order to return to its starting position. In 1928, at the age of 26, P. A. M. Dirac described the movement of an electron with an equation that united *quantum mechanics* and the *special theory of relativity*. In order to convince his students that Pauli's model is not so surprising, he took a wrench (instead of our ball in the middle of the room) connected to a chair with three strings, rotated the wrench for 4π radians and then untangled the strings without moving the wrench. To learn more about the calculus of spinors and the theory of angular momentum, see [**BL**, Ch. 2]. In the early thirties this was such a hot topic at Niels Bohr's Institute for Theoretical Physics that some games were even invented. The Danish poet, writer, and mathematician Piet Hein called his game *Tangloids*, and it has been played for many years in Europe [**G**, p. 28]. Each of the two players holds one piece of wood with three holes that are connected by three shoe laces, see Figure 5(a). One player rotates her/his piece around *any* axis for 4π radians and the other one tries to untangle the strings while both pieces are allowed only to translate (rather than rotate). Then they reverse the roles and the one who untangles the shoe laces faster is the winner. (Why is this a good model for Question 1?) There is another related puzzle [**BCG**], in which you have to braid a paper (Figure 5(b)) without tearing it and using glue. The underlying principle is most probably quite old and known to craftsmen in leather (scouts should know about it as well). Hint: start braiding at one end and then untangle at the other end.



(a)



(b)

Figure 5

By contrast, Question 2 concerns *nonexistence* and is therefore certainly in the domain of mathematics. It has been solved at least in two ways: Newman employed Artin's braid theory [**New**], cf. [**FB**], and Fadell used configuration spaces [**Fa**].

It seems that everything has already been done, except that there is no obvious connection between solutions of Question 1 and Question 2. One can still wonder why that graphical solution (Figure 4) works; checking it reminded us of a towing horse seeing only the path in front of itself. Could we understand it better or find a more natural solution, and use that to solve Question 2?* Let us use our

*At that time the author was not even aware that Question 2 had already been solved.

imagination (if not some childhood memories) and picture a girl/boy sitting at her/his desk, trying to satisfy the parents by studying some thick boring book. She/He soon gets tired and starts to play with a plastic model of an airplane. She/He wants to fly the plane forward on and on (planes do not fly backwards, remember?) while holding it firmly in one hand, however (s)he cannot move from the chair (the floor is squeaky and the parents might realize that (s)he is not studying any more), and (s)he wants to fly the plane without turning it upside down. She/He pretends to be a pilot of a commercial plane and passengers would get angry if (s)he did otherwise, at least those who do not use their seatbelts. Suddenly (s)he finds out that (s)he can move the plane along a path that has the shape of number 8. "Amazing", (s)he thinks, "if I moved the plane along a path in the shape of a circle my hand would tangle, and I would have to stop, but 8 works".

Could we use this ingenious solution in our problem? Well, let us twist and fold this figure of eight so that we see from the top only one "circle"; a mathematician would say that such an eight covers a circle (see Figure 6). The girl/boy has still no problem flying the plane in the path of this folded eight shape. Try it yourself; Figure 7 might be helpful (similar pictures have appeared in many journals and books, e.g., **[Ri]**, **[Str]**, **[Br**, p. 166**]**). But the girl/boy is now moving the plane in the path of a circle (for that reason we fold 8 completely), with the arm returning to the same position only after two full circles (during the first circle the elbow goes above the plane, while during the second circle it stays below the plane). There is no problem in making these circles so small that the plane is eventually just rotating. So the girl/boy can *rotate the plane in the same direction constantly while holding it firmly in the hand.* Actually, we can even say that (s)he can rotate the hand constantly in the same direction while it is turned up all the time. This phenomenon will be the leading idea of our proof of Question 2, and will contribute to finding a 'better' solution of Question 1. Are you puzzled? Think about it. If the plane were fixed, the hand would get twisted several times. So, in order to understand what is going on when we are moving elastic bands, we follow their twisting.

4. TWISTING NUMBER. Let us replace the line segments b, m, w with narrow bands. Such an approach is quite natural these days in knot theory and already has real-world applications. For example, when we study DNA molecules, which are long, thin, and usually double-stranded, we follow their twisting.

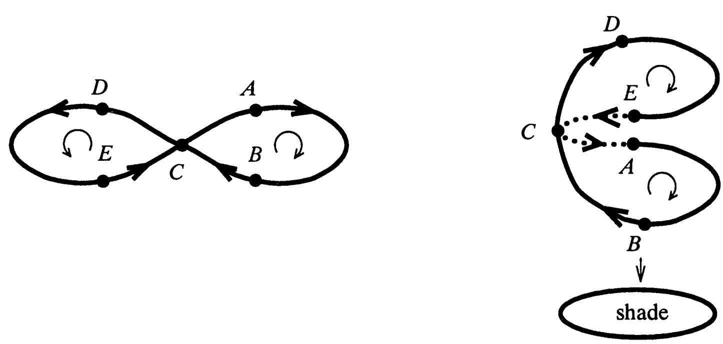


Figure 6

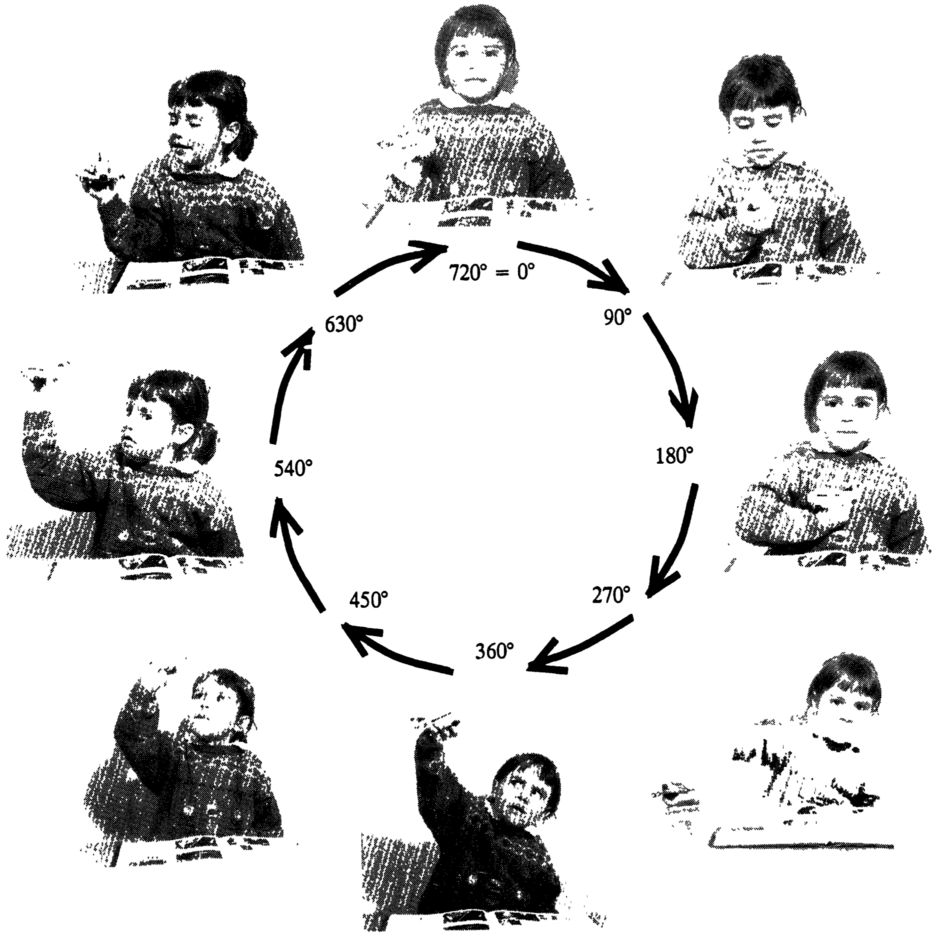


Figure 7. Eva Jurišić rotates plane while holding it firmly in her hand. For a similar demonstration by Uroš Jurišić, see http://crypto2.uwaterloo.ca:80/~ajurisic.

Before we attempt to solve Question 2 in this extended version (with bands) we study natural simplifications. The hand holding the plane could be considered as a single band. We try to find out what can happen to a single band in a three-dimensional ball (Lemma 1), and in a hollow ball H (Lemma 2). But first we introduce one of the simplest nontrivial invariants of ambient isotopy studied in knot theory. This is the linking number, defined for a pair of oriented curves in $\mathbb{R}^3$ (see Kauffman **[K1]** or **[K2]**). If a sign is associated to each crossing of the two-dimensional representation of the two curves as described in Figure 8, then the *linking number* is defined to be the sum of signs over all crossings of one closed curve with the other divided by two; signs of a curve crossing itself are not included. Since the

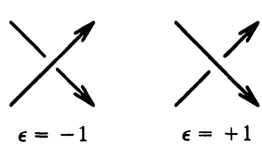


Figure 8. Right-hand-rule.

linking number is an ambient isotopy invariant, it does not depend on the two-dimensional representation we have chosen. In the nineteenth century, when knot theory was in its infancy, Gauss computed inductance (linking numbers) in a system of linked circular wires. The linking number plays an important role in the study of the effects of enzymes on a circular DNA, i.e., one that can be represented by a closed band in $\mathbb{R}^3$ whose boundary has two components [**Su1**].

Let X be a band of small width in a three dimensional subspace of $\mathbb{R}^3$ placed with its core (Figure 9(a)) along a line, with its ends in a plane, and both sides of the band oriented from the left end to the right end (Figure 9(b)). If a sign is associated again, as described in Figure 8, to each crossing of one side with the other in a two-dimensional representation in the plane that contains their ends, then the *twisting number*, $\mathrm{Tw}(X)$, is defined to be the sum of all signs divided by two. Intuitively, the twisting number is half the difference between the number of twists through the angle π and the number of twists through the angle $-\pi$.

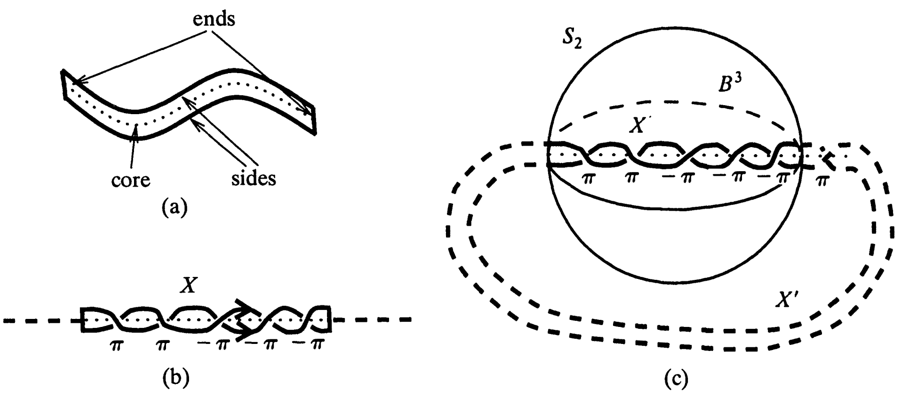
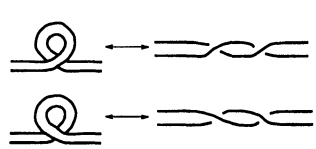


Figure 9

We could avoid relying on a two-dimensional representation, and give White's more general definition of this quantity by a line integral, which sums the amount of twisting along the core [**Su2**, p. 22], [**CS**]. However it would still not be an ambient isotopy invariant. Further, if our band were a closed band in three-dimensional space, then we could also define a writhing number, $\mathrm{Wr}(X)$, by orienting the core, assigning, as in the case of the linking number, a sign to each crossing, and then summing up all the signs. Then the sum of the writhing number $\mathrm{Wr}(X)$ and the twisting number $\mathrm{Tw}(X)$ equals the linking number $\mathrm{Lk}(X)$. This is J. H. White's Conservation Law [**W**]: $\mathrm{Lk}(X) = \mathrm{Wr}(X) + \mathrm{Tw}(X)$; see Figure 10.



one writhe = one twist

Figure 10

Lemma 1. *Let B^3 be a ball in $\mathbb{R}^3$ that has S_2 for its boundary. If X is a band of small width, placed along x-axis, and with its ends on the equator, then the twisting number of X cannot be altered by an ambient isotopy of B^3 fixing S_2.*

Proof: We join the ends of the band X in the exterior of B^3 with a band X' so that the boundary of $X \cup X'$ has two components; see Figure 9(c). Any ambient isotopy of B^3 fixing S_2 can be extended with the identity on the complement of B^3 to an ambient isotopy of $\mathbb{R}^3$. If the ambient isotopy of B^3 changed the twisting number of X, then the ambient isotopy of $\mathbb{R}^3$ would change the linking number of the boundary of $X \cup X'$, which is impossible. □

Hence, when we examine the twisting number of a band in a three-dimensional ball it suffices to follow only the sides of the band. This is not the case with a hollow ball. Let A be a band of small width from S_1 to S_2 in H, placed along z-axis and with its ends in the xz plane. There exists an ambient isotopy of H that alters the twisting number of A by two. This should already be clear from the girl/boy's game with the airplane, but it is even more transparent from Figure 11.

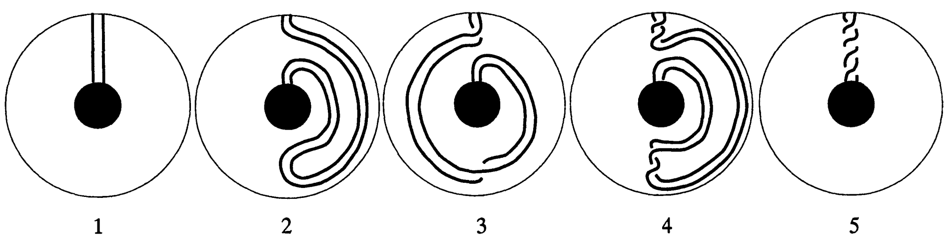


Figure 11. We get from the second figure to the third one by flipping the part of the band closer to the sphere S_2 in the upper hemisphere. The next step is flipping the same part of the band in the lower hemisphere.

The last position can be drawn in another way, see Figure 12.

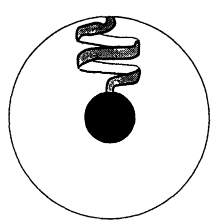


Figure 12

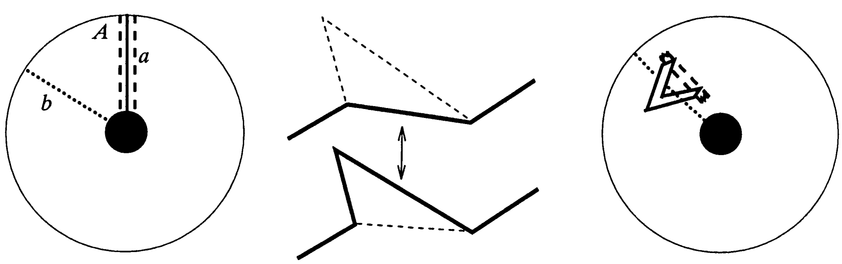
We now come to the heart of our problem. We are in a situation where we have to use a little bit of topology to prove the nonexistence part. However, due to the following two facts, we can stick with a combinatorial approach.

(1) Since our starting position can be drawn with polygonal lines, any position can be drawn with polygonal lines after a small deformation.
(2) If there is an ambient isotopy between two polygonal positions, then there exists a piecewise linear ambient isotopy between these positions [**BZ**, p. 4 and Corollary 3.16].

Lemma 2. *An ambient isotopy of H fixing S_1 and S_2 can alter the twisting number of the band A only by a multiple of two.*

There is evidently no ambient isotopy of the hollow ball H that alters the twisting number of the band A by one half, since the spheres S_1, S_2, and therefore also the ends of A, are fixed. The remaining part, namely that the twisting number cannot be altered by one, appears implicitly in **[K1**, VI.1, VI.18] without a proof; see the end of this section. We sketch our proof:

Proof: Let us denote by a and b the line segments from $(0,0,2)$ to $(0,0,1)$ and from $(2,0,0)$ to $(1,0,0)$ respectively (Figure 13(a)). So a is the core of the band A and we assume that it moves with an ambient isotopy of H. On the other hand, b will be considered as a ray, which does not move. As previously noted, we can assume **[BZ**, Prop. 1.10] that an ambient isotopy is piecewise linear, which implies that the induced movement of a is realized by a finite number of Δ-moves (Figure 13(b)) and that after every such move A and b are disjoint. It is important to note that as long as the band A does not go through the ray b, its twisting number cannot be changed, by Lemma 1. Let us now see what happens when during some Δ-move the band A goes through the ray b. We will show that we can move a to the same place, without going through the ray b, but at the cost of making two new twists of the band A; see Figure 14, which is obtained by slightly modifying Figure 11. Unfortunately we are not sure that we can perform the 'move' from Figure 14 in the hollow ball H, since some parts of the band A can be in the way. However, we can get rid of these difficulties, since we actually do not need the whole hollow ball H for that move. The part of the band on which we performed the Δ-move when taking the band A through b, lies at the outset in some hollow ball $H' \subset H$; see Figure 15(a). Assume that this hollow ball H' is also being deformed by our ambient isotopy; see Figure 15(b). But no matter how deformed, it will never



(a) (b) Delta move (c)

Figure 13

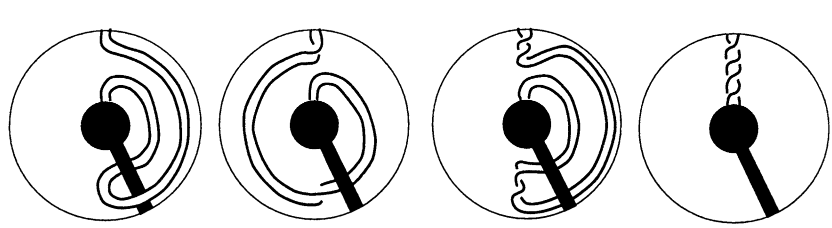
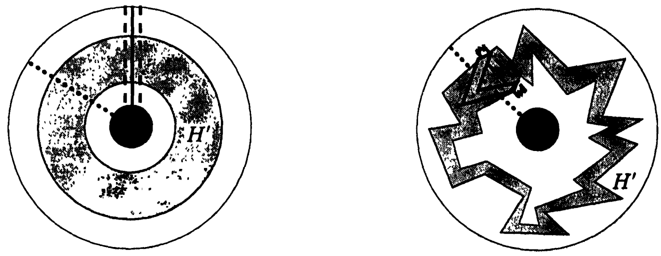


Figure 14

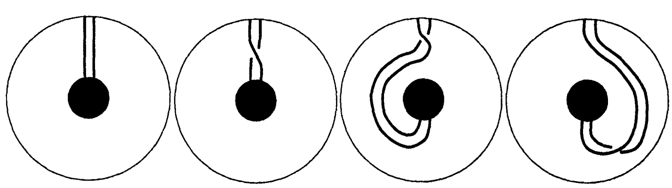


(a) (b)

Figure 15. Two plane sections.

contain any other part of the band A and it will always look like (i.e., be homeomorphic to) a hollow ball. So now we use this 'star-like' hollow ball H' to perform the move from Figure 14 backwards. Finally we can conclude that the twisting number of the band A can change only by a multiple of two. □

A reader who is anxious to see the solution of Question 2 can now proceed to the next section. But on the way let us mention an interesting connection with the quaternion group (see **[K1]** and **[K3]**). Let i, j, and k be the rotations of S_1 around the coordinate axes x, y, and z, respectively, through π radians. Let us define a group generated by i, j, k modulo the equivalence relation $\sim$ of an ambient isotopy in a hollow ball H with the band A keeping S_1 and S_2 fixed. Then Figure 11 shows that $i^4 = j^4 = k^4 = 1$. Similar pictures show that $i^2 = j^2 = k^2$, $ij = k$, $jk = i$, $ki = j$ and if we denote i^2 with minus one (-1) then also $ji = -k$, $kj = -i$, $ik = -j$. Figure 16 proves the last equality. Thus this group is exactly the quaternion group.



i ik $j^{-1} = j^3 = -j$

Figure 16

The nonexistence part of Lemma 2 is expressed in **[K1]** as $i^2 \neq 1$.

5. SOLUTION. We are now ready to apply Lemma 1 and Lemma 2 to Question 2. Let B, M, W be untwisted bands in H along line segments b, m, and w, respectively. Suppose there exists an ambient isotopy of H fixing S_1 and S_2 that wraps m k times around w, for some $k \in \mathbb{Z}$. If the width of the bands is small enough then this isotopy also wraps the band M k times around the band W and we have the position shown in Figure 17. If we orient the sides of the bands as in Figure 17, then, by Lemma 2, the twisting number of B is equal to $2n$, for some

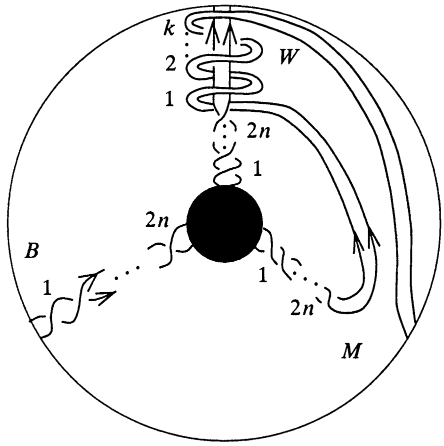


Figure 17

$n \in \mathbb{Z}$. Look at $B \cup W$. We can connect B and W by an untwisted band lying in the interior ball to form a single band with its ends on the boundary of the big ball. Then it follows from Lemma 1 that the twisting number of W is $-2n$. A similar argument can be applied to $B \cup M$: since we can pull M straight (moving it through W, of course) without introducing any new twists, the straight part of M has $-2n$ twists. On the other hand, when dealing with $M \cup W$ we are not allowed to move M through W (or itself or the interior ball). In order to pull M straight we first get rid of coils of M by using the move described in Figure 14 exactly k times (moving M through B, of course). Pulling M straight introduces $2k$ additional twists in M. Therefore the twisting number of M equals $-2n + 2k$. But on the other hand, by Lemma 1 for $M \cup W$, it equals $2n$. Hence $k = 2n$. Therefore there is no ambient isotopy of H fixing S_1 and S_2 that wraps M an odd number of times around W.

6. CONCLUSION. If we apply the result of Question 2 to the core and the sides of the band A in H, we get the nonexistence part of Lemma 2. Therefore we conclude that the result of Question 2 and the nonexistence part of Lemma 2 are equivalent.

Let us now give a more natural solution of Question 1. First we can move the elastics from the starting position in Figure 3(a) to the position in Figure 18. Second, we repeat the move from Figure 11 for the parallel parts of m and w in order to wrap m twice around w. Now it remains to untangle b with the "parallel" parts of m and w (remember that we can always untangle two elastic strings from S_1 to S_2).

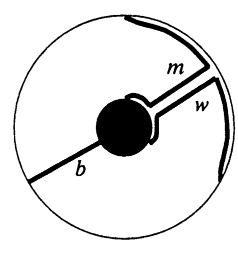


Figure 18

Finally we explain the phenomenon mentioned at the beginning. Suppose you coiled a long extension cord around your left shoulder. Imagine a three-dimensional ball in place of your shoulder, with the North Pole pointing in the direction of your left hand. So the cord gets coiled along the equator from the direction of the North Pole; see Figure 19(a). After the cord is stored for some time, we usually do not remember where the North pole is, and we unwind it either in the direction of the North or the South Pole. In the first case the cord will not be twisted, while in the second case (when we start to uncoil the cord straight from the wall where we stored it, see Figure 19(b)) Figure 11 guarantees that it will be twisted many times. Note that if you perform this experiment the cord either has to be long enough or you should fix one end and hold the other one with your hand. Lawn mowers, who are aware of the twisting problem, coil their extension cords in the shape of number 8, cf. Figure 6 and Figure 10. In industry, on the other hand, long extension cords are coiled on rolling cylinders.

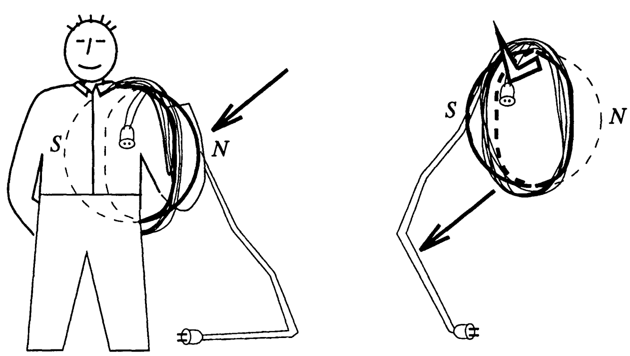


Figure 19

Let us conclude with applications from **[Sto]**. Figure 11 can be redrawn one more time (see Figure 20) in order to demonstrate that when the cord (in the shape of a question mark) goes around the small sphere once and the small sphere is rotated through 4π radians in the same direction, the cord resumes its original position. D. A. Adams used this fact to build a device that can supply electric power to a rotating platform through flexible wires (i.e., one that prevents the wires from twisting up and breaking). The reverse approach has actually been used widely in the electrical manufacturing industry, for example, to twist electrical wires uniformly. Many similar but simpler machines were discovered later by physicists **[Ri]**.

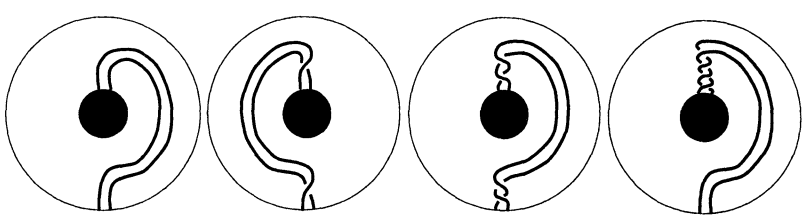


Figure 20

We can push the discrete approach further. Suppose we want to record a specimen on a rotating platform, but we do not want the viewer of our recording to notice the rotation. Can you find a solution where the camera does not move around? (Hint: Replace the band in Figure 20 with a few prisms.) A solution is shown in Figure 21. The same principle is used in a *periscope*, i.e., an instrument by which an observer obtains an otherwise obstructed view, to be able to look around while not moving her/his head; see **[BW**, pp. 243–244].

We hope that some teachers will find useful material for entertaining students while introducing the quaternions, for example. For further reading on related subjects see **[Re]**, **[K1]**, **[K2]**, **[K3]**, **[Br]**, **[BZ]**, **[Sti]** for knot theory, **[Su1]**, **[Su2]** for applications in studies of DNA, and **[W]**, **[CS]** for theoretical background on linking, twisting, writhing, and winding numbers.

APPENDIX. A topological solution of the first question is very well known to many topologists and can be found in many places, e.g., **[Br]**. Here we present the solution outlined by Vrabec.

This time we assume that the elastic bands b and w are along the x-axis. Let us define a loop u: $[0,1] \to SO(3)$ by

$$u(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi t & -\sin 2\pi t \\ 0 & \sin 2\pi t & \cos 2\pi t \end{bmatrix};$$

we can think of u as a 'continuous' rotation of $\mathbb{R}^3$ for 2π radians around x-axis. The concatenation $u * u$ is then a rotation through 4π radians. The action of the set $\{(u * u)(t)\colon 0 \le t \le 1\}$ on $P = (0,1,0)$ gives us the mapping (loop) $[0,1] \to S^2$, $t \mapsto [(u * u)(t)](P)$. The graph of this mapping is a path in $[0,1] \times S^2$. If we identify this product with the hollow ball in a natural way, the graph of the mapping mentioned above corresponds to a path that is wrapped twice around the x-axis. It is well known that $SO(3) \cong RP^3$ and so $\pi_1(SO(3)) = \pi_1(RP^3) = \mathbb{Z}_2$. Therefore there exists a homotopy between $u * u$ and $[0,1] \to \{\mathrm{id}\} \subset SO(3)$; the equivalence class of u is actually the only nontrivial element of the fundamental group of $SO(3)$. So let

$$\{F_s\colon [0,1] \to SO(3)\colon 0 \le s \le 1\}$$

be the corresponding homotopy, $F_0(t) = \mathrm{id}$, and $F_1(t) = (u * u)(t)$ for each t. From this we get the ambient isotopy (family of homeomorphisms)

$$\{G_s\colon [0,1] \times S^2 \to [0,1] \times S^2\colon 0 \le s \le 1\}$$

of the hollow ball $[0,1] \times S^2$ in the following way: $G_s(t,Y) := (t,[F_s(t)](Y))$ for $t \in [0,1]$, $Y \in S^2$. Evidently we have $G_0 = \mathrm{id}$ and G_1 is the mapping that maps $[0,1] \times \{P\}$ to the path mentioned above that is wrapped twice around the x-axis. The ambient isotopy G pointwise fixes the boundary spheres, since $G_s(i,Y) = (i,Y)$ for $i \in \{0,1\}$ and $Y \in S^2$, i.e., $F_s(i) = I_3$ (the identity matrix 3×3), which follows from the fact that F_s is the homotopy of the loops with starting and ending points at $\mathrm{id} \in SO(3)$. Finally $G_1(t,(x,0,0)) = (t,(x,0,0))$, since $F_1(t) = (u * u)(t)$ is identity at $(x,0,0)$ (i.e., rotations around the x-axis fix points on the x-axis), so after homotopy is applied b and w are at their starting places. □

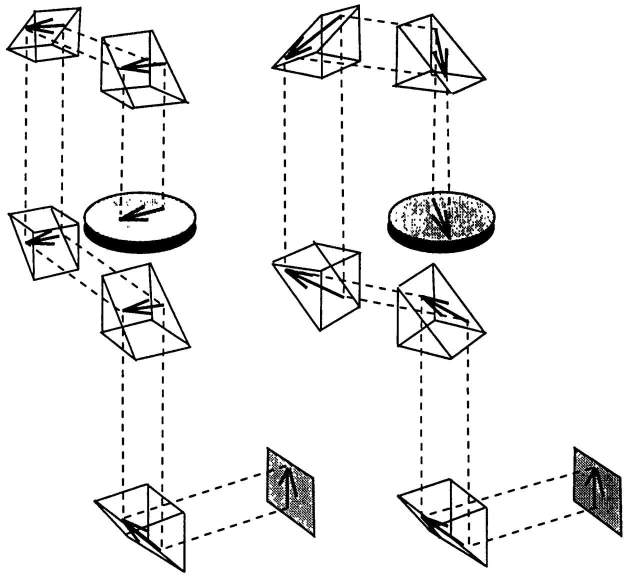


Figure 21. Train of prisms 'untwists' an optical beam.

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It's De-Lovely

(With apologies to Cole Porter)

You take two sets, you form their meet,
Complementation is quite a treat,
It's delightful, It's DeMorgan, it's d'algebra

You grab a 'zee' in the complex plane,
Taking its power is not a pain,
It's delicious, it's DeMoivre, it's d'analysis.

We can tell at a glance
Why the math folks sing "Vive la France ,
We can swear their view of logic's
Gotta be true, "true as l'bleu".

But don't take sides, just to have some fun,
All mathematics is one-to-one,
It' delightful, it's delicious,
It's de-stributive, it's dee-'wye'-dee-'ex',
It's d'limit, it's de-luxe, it's de-lovely.

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