If $p > 0$ and $n > 2$, equality holds if and only if (*) of Theorem 2 holds.

As in the case of (3), equality always holds in (6) (and hence in (5)), if n is 1, 2 or 3. Of course equality always holds in (6), if $p = 0$.

The author wishes to thank Professors D. Gale and D. J. Newman for their valuable suggestions.

References

1. H. D. Block, Discrete analogues of certain integral inequalities, Proc. Amer. Math. Soc., 8 (1957) 852–859.

2. H. G. Eggleston, Convexity, Cambridge Tracts in Mathematics and Mathematical Physics, No. 47, Cambridge University Press, 1958.

3. K. Fan, O. Taussky, and J. Todd, Discrete analogs of inequalities of Wirtinger, Monatsh. Math., 59 (1955) 73–90.

4. W. Fenchel, Über Krümmung und Windung geschlossener Raumkurven, Math. Ann.,101 (1929) 238–252.

5. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, 2nd edition, Cambridge University Press, 1952.

6. A. M. Pfeffer, On certain discrete inequalities and their continuous analogs, J. Res. Nat. Bur. Standards Sect. B, 70B (1966) 221–231.

7. I. J. Schoenberg, The finite Fourier series and elementary geometry, this MONTHLY, 57 (1950) 390–404.

CURRENT TRENDS IN ALGEBRA

GARRETT BIRKHOFF, Harvard University

1. Introduction. Symbolic algebra is much older than many mathematicians suppose; it can be traced back at least to Diophantus of Alexandria (ca. 250 A.D.) and Brahmagupta (ca. 598–665 A.D.). For this early work, see Cajori [7, Arts. 101–5] and Ball [1, pp. 154–6]. Even so-called "modern" algebra is over a century old!

Garrett Birkhoff is the Putnam Professor of Pure and Applied Mathematics at Harvard, where he did his undergraduate and graduate work, was a Junior Fellow in the Society of Fellows, and has served on the faculty since. He has been a Visiting Lecturer at the University of Washington, University of Cincinnati, and the National University of Mexico, and held a Guggenheim Fellowship. He has served as President of SIAM, Vice-President of the AMS, the MAA, and the American Academy of Arts and Sciences, and Chairman of the CBMS. He is a member of the American Philosophical Society and the National Academy of Sciences, and has received honorary degrees from the National University of Mexico, the University of Lille, France, and the Case Institute of Technology.

His extensive publications in modern algebra, fluid mechanics, numerical analysis, and nuclear reactor theory include the books *Hydrodynamics* (Princeton University Press, 1950); *Lattice Theory* (American Mathematical Society Colloquium Publications, 1940, Third Edition 1967); *Survey of Modern Algebra* (with S. Mac Lane, Macmillan, 1941, 1953, 1965); *Jets, Wakes, and Cavities* (with E. H. Zarantonello, Academic Press, 1957); *Ordinary Differential Equations* (Ginn, 1962); *Algebra* (with S. MacLane, Macmillan, 1967); *Modern Applied Algebra* (with T. C. Bartee, McGraw-Hill, 1970). *Editor.*

When you realize this, you should not find it too hard to believe that the availability of high-speed computers is giving rise to new trends in algebra. My ultimate aim is to sketch for you, in §§8–10, what I think these trends are. But I wish to lead up to this theme by a brief résumé of the development of algebra as we know it today, over the past several centuries.

2. Classical algebra. The name *modern algebra* was originally intended (in 1930) to signify a contrast with *classical algebra*, which was generally understood to mean the *theory of equations*. This may be defined as the art of solving *numerical problems* by *manipulating symbols*, and seems to have originated with Al-Khwarizmi and other Islamic mathematicians during the period 800–1000 A.D. As we know, its most essential idea consists in replacing each verbal statement about *numerical* quantities by a symbolic *equation*, whose terms can be rearranged and combined by well-established general laws to give a sequence of equivalent but, hopefully, simpler equations. The original equation can be considered as "solved" when the unknown quantity has been isolated on one side of the equality symbol $=$, on the other side of which is some expression involving only known quantities.

Though the word "root" (of an equation) can be traced back to the Sanskrit,[1,2] and the word "power" (of a number) appears in al-Khwarizmi's *Algebra* ("al-jabr"), development of classical algebra in its present form was very gradual. The "al-jabr" of the Arabs did not become widely used in Western Europe, and the symbols $+$ and $-$ did not achieve their present significance, until nearly 1500. A major advance followed shortly thereafter, the solution of cubic and quartic equations by radicals being already contained in Cardan's *Ars Magna* (1545).

For the next two centuries, progress in algebra was mainly[3] in connection with its applications: to (analytic) geometry, which gave a vivid meaning to negative numbers, and to the calculus through the use of infinite series. Until after 1750, the significance of imaginary roots and complex numbers remained quite obscure, and even discussions of simultaneous linear equations and determinants were unsystematic and fragmentary.

But from 1750 to 1830, thanks especially to the work of Euler, Lagrange, and Gauss, classical algebra developed rapidly into approximately its present form. Thus the exponential function e^z became defined for all complex z as a power series, and as a result $a^z = e^{z \ln a}$ became well-defined for all positive a and complex z. The "Lagrange resolvent" was also invented by Euler [**13**, p. 27].

Above all, the Fundamental Theorem of Algebra was recognized as such, clearly stated, and proved. Euler considered its *real* forms, whose equivalence is easily shown. Two of these are:

(a) Every real polynomial of degree $n > 2$ has proper factors.

(b) Every real polynomial can be (uniquely) factored into real linear and quadratic factors.

1. All notes are collected together at the end of the paper.

Condition (a) follows for $n < 5$ from Cardan, and for $n = 5$ because every real polynomial of odd degree has real roots. Euler satisfied himself that it was true for all n, but his proof is obscure.

Conditions (a) and (b) are easily shown to be equivalent also to the usual statement of the Fundamental Theorem of Algebra:

(c) Every complex polynomial can be factored into linear factors.

Gauss gave many relatively rigorous proofs of (c) from about 1800 on, and made it clear that all polynomial equations had solutions in terms of complex numbers $x + yi$, $i = \sqrt{-1}$, while the geometrical interpretation of complex numbers as points in the (x, y)-plane gave them a more than symbolic meaning.

Gauss also developed systematic iterative as well as elimination techniques for solving systems of simultaneous linear equations, and the laws of determinants also became generally known—all by 1825 or so.

A few years later, Galois and Abel showed that it was impossible to solve a general equation of the fifth degree by radicals[4], and after this mathematicians gradually began to turn their attention from the theory of equations to *non*-numerical applications of symbolic algebra (e.g., to groups, vectors and matrices).

MODERN ALGEBRA TO WORLD WAR I

3. "Modern" algebra to 1860. As a result, although real and complex algebra dominated the textbook literature for a full century after 1830, "modern" algebra had already achieved some notable successes by 1860.

Actually, already by 1770, Lagrange was interested in the "symmetric group" of all permutations of n letters and its subgroups, whose relevance to the solution by radicals of the general polynomial equation

$$x^n + a_1x^{n-1} + \cdots + a_n = (x - x_1)(x - x_2)\cdots(x - x_n) = 0, \tag{1}$$

he clearly recognized. A by-product of this interest was the Lagrange Theorem, that the order of any subgroup of a group G divides the order of G.

Ruffini, Galois, and Cauchy made further contributions to the development of group theory before 1845 [**13**, pp. 45–53]; Galois also made (in 1830) a fundamental contribution to the theory of fields, by constructing a *finite field* of each prime-power order p^r. (For formal definitions of groups and fields, see §3.)

Somewhat earlier Legendre and Gauss (1801) had initiated the study of *commutative rings*, by constructing the ring Z_n of the integers "modulo" n (i.e., in which integral multiples of n are set equal to zero) and the ring $Z[i]$ of all "Gaussian integers" $m + ni$, where $m, n \in Z$ are ordinary integers[5] and again $i = \sqrt{-1}$. Moreover Gauss, had proved that factorization into primes was *unique* in $Z[i]$.

By 1850, *noncommutative rings* were also being studied. Thus Hamilton introduced his *quaternions* in 1843; since they contain the complex numbers as a special case, they may be called *hypercomplex numbers*. And in the first edition of

his book *Ausdehnungslehre* (1844) H. Grassmann discussed both *vector algebra* (a fairly natural generalization of Descartes' symbolic method for treating geometry) and, somewhat vaguely, hypercomplex numbers in general. These concepts were made much more precise (and their connections with n-dimensional geometry clarified) by Cayley; by Hamilton in the Preface to his book on *Quaternions* (1859); and by Grassmann in the second edition of his book (1878). Moreover, Cayley showed in 1858 [**13**, p. 84] that the theory of determinants of Vandermonde and Laplace was only one aspect of a much more powerful *matrix algebra*. Matrix algebra is much like ordinary algebra, except that for general matrices A and B, $AB \neq BA$: the multiplication of matrices, like that of quaternions, is *non*-commutative. Indispensable for all pure and applied mathematicians today, matrices were first introduced formally by Cayley in 1858, and gradually revolutionized linear algebra.[6]

Shortly before, two other novel areas of modern algebra had been opened up. In 1854, Boole had published his *Introduction to the Laws of Thought*, in which he showed that a substantial part of Aristotelian logic was described by an analog of ordinary algebra now called "Boolean algebra." This novel "algebra of logic" satisfied not only most of the laws of ordinary algebra, but also the curious identities

$$a^2 = a + a = a \text{ (which today would be written } a \wedge a = a, \vee a = a),$$

$$(a + b)a = a, \text{ and } (a + b)(a + c) = a + bc.$$

4. The axiomatic approach. We have just seen that many of the major branches of so-called "modern algebra" (rechristened "the new math" by the popular press in the post-Sputnik era) were already known to mathematicians by 1860. However, the axiomatic approach to the foundations of algebra did not come until later. Lagrange derived the Lagrange theorem for groups and Galois constructed Galois fields without ever thinking of groups or fields as defined by postulates at all; their assumptions were entirely intuitive! Even the names "commutative" and "distributive" for the corresponding laws of manipulation were not introduced (by Servois) until 1814,[7] nor the term "associative" (by Hamilton) until 1835.

The emancipation of algebra from exclusive concern with the real and complex fields owes much to the philosophical speculations about algebra of Peacock, Woodhouse,[8] Hamilton, de Morgan, Boole, and Cayley, but E. T. Bell's claim [**3**, pp. 180–1] that it was Peacock who: "first perceived common algebra as an abstract hypothetico-deductive science of the Euclidean pattern" goes too far. Though Peacock anticipated Hankel in announcing the "principle of permanence of equivalent forms," his "Symbolical Algebra" is mainly concerned with geometrical applications, and does not even mention axioms or postulates. In these qualities it resembles H. Grassmann's *Ausdehnungslehre* (1844).[9]

The role of axioms emerges much more clearly from the *Formenlehre* of R. Grassmann (1872); the *Operationskreis der Logikkalkul* of E. Schröder (1877); the axiomatic treatments of groups, fields, modules, and ideals by Cayley (1878), Frobenius

and Stickelberger (1879), Dedekind,[10] Weber (1882, 1893), and E. H. Moore; and the independent contemporary work of Benjamin Peirce and his son, C. S. Peirce, at Harvard (1870–1881).[11]

Influenced by these writings, Peano[12] initiated in 1888 his axiomatic approach to arithmetic, about which I shall say much more later. A decade later, in his *Grundlagen der Geometrie* [9], Hilbert tried to improve on Euclid. He succeeded from the standpoint of rigor, but not from that of pedagogy! Perhaps his most fundamental contribution to axiomatics was his clear formulation of the notions of independence, consistency and completeness for axiom systems.

In 1902, E. H. Moore showed that Hilbert's own axioms were not independent, and during the next ten years E. V. Huntington, L. E. Dickson, and O. Veblen made other painstaking analyses of the independence of postulate systems for groups, fields in general, the real and complex fields in particular, the algebra of logic, and the foundations of geometry. One can get an excellent picture of this work by reading the papers by Moore and Huntington;[13] for a more colorful if less reliable survey, see [2, Ch. 3].

Partly as a result of such papers, the postulational approach to algebra finally became standard. Mathematicians found that amazingly few and simple postulates, many fewer than those of Euclidean geometry,[14] could provide a sufficient basis for very extensive algebraic theories. For example, all of group theory can be derived from general principles of logic and the following postulates, due to E. V. Huntington (1906).

DEFINITION. A *group* G is a set of elements (to be denoted by small Latin letters), any two of which, say x and y, have a *product* xy which satisfies the following conditions:

G1. Multiplication is *associative*: $x(yz) = (xy)z$ for all $x, y, z \in G$.

G2. For any two elements $a, b \in G$, there exist $x, y \in G$ such that $xa = b$ and $ay = b$.

(We have used Peano's notation $x \in G$ above; it signifies that "the element x is a member of (belongs to) the set G.")

Ingenious arguments can be used to deduce from these postulates various other simple conditions, for example, that: (i) any group G contains a unique "idempotent" element e satisfying $ee = e$, (ii) this element satisfies $ex = xe = x$ for all $x \in G$ (acts as an "identity" for G), (iii) the elements x and y in G2 are uniquely determined by a and b, and so on.

Similarly, the entire theory of fields can be deduced from the following set of postulates, also due to Huntington.

DEFINITION. A *field* is a set F of elements, any two of which have a *sum* $x + y$ and a *product* xy which satisfy the following conditions:

F1. Addition and multiplication are *commutative*:

$$x + y = y + x \text{ and } xy = yx \text{ for all } x, y \in F.$$

F2. Addition and multiplication are *associative*:

$$x + (y + z) = (x + y) + z \text{ and } x(yz) = (xy)z, \text{ all } x, y, z \in F.$$

F3. Multiplication is *distributive* on sums:

$$x(y + z) = xy + xz \text{ for all } x, y, z \in F.$$

F4. For any $a, b \in F$, there exists some $x \in F$ such that $a + x = b$.

F5. If $a + a \neq a$, then there exists some $y \in F$ such that $ay = b$. (Actually, Huntington weakened F5 by adding the condition $b + b \neq b$ to its hypothesis.)

(Of course, the hypothesis $a + a \neq a$ is just an indirect way of assuming that $a \neq 0$, necessary here because Huntington wanted to avoid assuming the existence of a "zero" 0 in F.)

The postulational approach to algebra, combined with an awareness of the relevance of all kinds of algebraic systems, stimulated an interest in enumerating *all possible algebraic systems* satisfying specified conditions: all finite fields (Galois had found them all), all groups of given order n, and so on. In this enumeration, one must of course identify all groups (or fields) which are *isomorphic*, that is, whose elements are related by a *bijection which preserves group multiplication* (in fields, which preserves addition *and* multiplication). Such a bijection is called an *isomorphism*.

5. Morphisms and subalgebras. More generally, it is helpful to know when two algebraic systems A and B are related by a (homo)*morphism*, or mapping θ: $A \to B$ which preserves all their defining operations. Finally, it is helpful to recognize the *subalgebras* of A, i.e., the subsets S of A which satisfy all postulates; under these circumstances, A is conversely called an *extension* of S. (Thus the complex field is an extension of the real field.) To test for being a subalgebra, it is usually sufficient to test for *closure* with respect to suitable operations. In a group, for example, a subgroup must contain: (i) the identity, (ii) with any x also x^{-1}; and (iii) with x and y also xy. In fields, one must require closure under addition, subtraction, multiplication, and division.

The preceding concepts apply to all of the usual kinds of algebraic systems; I shall come back to them in my next lecture.

6. Some deeper developments: 1860–1914. During the same decades that its foundations were being clarified by the postulational method, the scope and depth of algebra grew enormously. I can only indicate very sketchily a few especially remarkable results here.

First, Galois theory became clarified as follows; I shall stick to extensions of the rational field Q to fix ideas, but the results generalize to extensions of any field. Let $F = Q[x_1, \cdots, x_n]$ be the field generated by the roots of a polynomial

$$p(x) = (x - x_1)(x - x_2)\cdots(x - x_n) = x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with coefficients $a_k \in Q$. Define the *Galois group* $F(f: Q)$ of F (and of $p(x)$) over Q to be the group of all automorphisms α of F such that $\alpha(x) = x$ for all $x \in Q$. Then the theorem of Galois states that the equation $p(x) = 0$ is solvable by *radicals* in terms of the coefficients (i.e., over Q) if and only if the Galois group $G(F: Q)$ is "solvable" in the following sense.

DEFINITION. Define a *composition series* of a finite group G to be a chain of subgroups of G,

$$1 < S_1 < S_2 < \cdots < S_r = G,$$

each of which is a maximal normal subgroup of the one following. Form the associated quotient-groups S_k/S_{k-1}. Then G is called *solvable* when these quotient-groups are all *Abelian* (it is equivalent that they all be of prime order).

Second, pure group theory acquired depth. Among the many remarkable theorems about finite groups proved in the half-century 1860–1914, I shall mention only a few. First, it was shown that the set of S_k/S_{k-1} is the same (up to isomorphism and rearrangement) for all composition series (Jordan-Hölder Theorem). Again, it was shown that any group of prime-power order p^n is solvable. Finally, it was shown that if p^n divides the order of a group G, then G has a subgroup of order p^n (Sylow theorem).

Third, in the area of algebraic number theory, Dedekind developed ideal theory, and applied it to generalize the pioneer result of Gauss on the unique factorization of Gaussian integers, to a sweeping unique factorization theorem for any algebraic number field (i.e., any subfield of the complex field C having finite linear dimension over the rational subfield Q). Namely, he showed that factorization into prime ideals is unique.[15]

Dedekind's deep interest in ideal theory and in unique factorization into primes also led him to consider the operations of greatest common divisor (g.c.d.) and least common multiple (l.c.m.) from a postulational standpoint. Recognizing their analogy with "and" and "or" in Boolean algebra, he was led to develop and apply the elementary theory of *lattices* ("*Dualgruppen*"), *modular lattices*, *distributive lattices*, and *vector lattices* in two pioneer papers (1897, 1901), thus founding a major new branch of algebra which contained Boolean algebra as a special case.

7. Linear associative algebras. In 1870, at about the same time that Dedekind was developing ideal theory into a powerful tool, Benjamin Peirce of Harvard made a pioneer study of the systems of "hypercomplex numbers" vaguely adumbrated by Grassmann, Hamilton and Cayley. Peirce began by defining a "linear algebra" over a field F as a set A whose elements are arbitrary linear combinations

$$\mathbf{a} = (a_1, \cdots, a_r) = a_1\mathbf{i}_1 + \cdots + a_r\mathbf{i}_r \tag{2}$$

of r basis elements $\mathbf{i}_l$, multiplied by some rule of the form

(2′) $$\mathbf{a}\cdot\mathbf{b} = (\Sigma a_l b_m)\mathbf{i}_l\cdot\mathbf{i}_m = \Sigma a_l b_m \gamma_{lmn}\mathbf{i}_n;$$

the constants γ_{lmn} can be any scalars (elements of F). He called a linear algebra *associative* when the multiplication defined by (2′) is associative.

A very notable linear associative algebra is provided by Hamilton's quaternions, which have four basic elements $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and hence 64 constants (mostly zero) defined by the rules

(3) $$\mathbf{1}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{1} = \mathbf{a} \text{ for all } \mathbf{a}, \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$

(3′) $$\mathbf{i}\cdot\mathbf{j} = -\mathbf{j}\cdot\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\cdot\mathbf{k} = -\mathbf{k}\cdot\mathbf{j} = \mathbf{i}, \quad \mathbf{k}\cdot\mathbf{i} = -\mathbf{i}\cdot\mathbf{k} = \mathbf{j}.$$

The identities of (3′) are clearly those for vector products. The quaternion algebra $R[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ over the real field is also a *division algebra*: any nonzero quaternion $\mathbf{a} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \neq \mathbf{0}$ has an inverse, given by

(3″) $$\mathbf{a}^{-1} = (a_0 - a_1\mathbf{i} - a_2\mathbf{j} - \mathbf{a}_3\mathbf{k})/(a_0^2 + a_1^2 + a_2^2 + a_3^2).$$

Peirce[16] showed that the complex numbers and the quaternions formed the *only* hypercomplex division algebras over the real field.

Even more important is the *full matrix algebra* $M_n(F)$ of all n^2-matrices $A = \| a_{lm} \| = \Sigma a_{lm} e_{lm}$. The basis elements e_{lm} of $M_n(F)$ are multiplied by the rules that

(4) $$e_{lm}e_{l'm'} = \begin{cases} e_{lm'} & \text{if } m = l' \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the constants are given (in a slightly changed notation) by

(4′) $$\gamma_{lm,l'm',l''m''} = \begin{cases} 1 & \text{if } l' = m, l'' = l, m'' = m' \\ 0 & \text{otherwise.} \end{cases}$$

From 1870 on, many mathematicians tried to classify linear associative algebras over the real and complex field, using the Fundamental Theorem of Algebra as a tool where convenient. Papers by Frobenius (1878, 1903), Molien (1893), and Cartan (1898) were especially noteworthy.[17]

In a remarkable paper published in 1907, Wedderburn showed that most of the structure theorems of Cartan and Frobenius could be proved for linear associative algebras over an arbitrary field! In particular, he proved the following basic results, whose precise meaning will be explained below. For further details, see [3, Ch. 11]. Wedderburn himself stated that "Most of the results contained in the present paper have already been given, chiefly by Cartan and Frobenius, for algebras over the rational field."

(i) Any linear associative algebra is the direct sum (in the vector space sense) of a "semisimple" subalgebra and a unique "nilpotent" invariant subalgebra;

(ii) The semi-simple summand in (i) is the direct sum of "simple" linear associative algebras, in a unique way;

(iii) Each *simple* summand in (ii) is, for some n, the "full matrix algebra" $M_n(D)$ of all $n \times n$ matrices $A = \| a_{ij} \|$ with entries a_{ij} in a suitable "division algebra" D over F, the field of scalars of the original linear associative algebra.

To explain (i), we recall that a linear algebra is called "nilpotent" when, for some finite integer n, all products $a_1 a_2 \cdots a_n$ of n factors vanish. A "subalgebra" of an algebra is a subset closed under addition and multiplication (as well as linear combination over the field of scalars); such a subalgebra K is called "invariant" when $k \in K$ implies $ak \in K$, and $ka \in K$ for any element a, even if not in K; this is the condition that K be an *ideal* in the sense of ring theory.

Not all linear algebras are associative. The most important family of non-associative algebras is the family of *Lie algebras*. In these, the associative law is replaced by the following three identities:

$$[aa] = 0,\ [ab] + [ba] = 0,\ [[ab]c] + [[bc]a] + [[ca]b) = 0,$$

true for all a, b, c. In the 1870's, Lie had shown that real and complex Lie algebras provided the key to the understanding of *continuous groups* based on a finite number of parameters. It was therefore most remarkable that Killing (1888–1890) and Élie Cartan (1894), were able to prove that Lie algebras satisfied structure theorems somewhat analogous to those for associative algebras stated above—and to determine all "simple" Lie algebras over C. This work of Killing and Cartan on the structure of Lie algebras came *before* the analogous work of Molien and Cartan on linear associative algebras[17a].

One can, perhaps, summarize the preceding developments in the statement that more was known about "modern algebra" by research algebraists in 1914 than most Ph.D's know today. However, algebra was still regarded as subordinate to classical analysis, and the complex field reigned supreme. Thus, of the two advanced texts on algebra (as distinguished from number theory) most widely used in 1900, Weber's began with a chapter on algebraic functions and Serret's with one on continued fractions!

8. Symbolic logic to Gödel. In retrospect, it seems not too surprising that the dramatic successes of 19th century algebraists and logicians should have encouraged some imaginative mathematicians to develop a *symbolic logic* which would reduce all theorem-proving to mechanical symbol manipulations according to prescribed rules or "axioms." Actually, this idea goes back at least to Leibniz, who envisioned around 1700 symbolic methods capable of "increasing the power of reason far more than any optical instrument has ever aided the power of vision." To his fertile mind, the powerful symbolic algebra of the differential and integral calculus (much of which he invented) must have seemed a direct confirmation of the potentialities of symbolic methods.

The symbolic approach was developed tremendously by Peano from 1889 on. His main contributions to it may be found in his *Formulario Matematico* (5th ed.,

1908), whose preface states that: "All progress in mathematics is in response to the introduction of symbols (ideographic signs). ... Among two symbolic systems, the one with fewer symbols is, in general, preferable. But the fundamental use of [symbolic methods] is to facilitate calculation." The preface continues with a review of the origin of various symbols, including $+$, $\times$, D (derivative), $\int$, and those for vector and Boolean algebra. It then proposes for general adoption the symbols $\in$ (for membership) and $\exists$ (there exists). Peano claims that with these, and a handful of other symbols and symbolic conventions, all mathematics can be presented in symbolic form.[18]

Actually, Peano was not the first mathematician to conceive of a purely symbolic mathematics. In 1634, Hérigone had written in the Preface of his *Cursus Mathematicus*: "I have invented a new method of making demonstrations, brief and intelligible, without the use of any language," and his symbolic style was adhered to by Wallis (1656) and Barrow (1655, 1660).[19]

Peano then substantiates his claim by 386 pages of text containing symbolic synopses of: (1) Mathematical Logic, (2) Arithmetic, (3) Algebra, (4) Vectors ("Geometry"), (5) Limits, (6) Derivatives, (7) Integrals. Most successful are Parts (1) and (2); the latter contains Peano's celebrated construction[20] of the nonnegative integers by his "successor function": $1 = 0+$, $2 = 1+$, $3 = 2+, \cdots$, and his derivation of the laws of arithmetic from it is superb. In 70 additional pages, Peano extends his purely symbolic treatment to plane curves, differential equations, and various other topics.

However, Peano's *Formulaire* must be viewed as primarily a thought-provoking *tour de force*, in spite of its wealth of ideas and insights. Nowhere does he list the rules of symbol-manipulation for passing from one formula to the next; he fails to provide a system of axioms for logic. His proofs, like Euclid's in geometry, can only be verified by attributing *meanings* to words.

This major gap was filled by Whitehead and Russell in their three volume masterpiece *Principia Mathematica* [**18**]. Here they specified carefully the symbol-manipulations ("rules of inference") which can be used infallibly in passing from hypotheses to conclusions in symbolic logic (mathematical reasoning).

Using their specified rules of inference as "axioms" for symbolic logic, Whitehead and Russell showed that one can paraphrase symbolically at least the construction of the real field R from the positive integers, as well as much of set theory and arithmetic. These major achievements were presented as empirical evidence supporting the thesis that *all mathematical theorem-proving can be reduced to mechanical symbol-manipulations* (i.e., to pure symbolic logic).[21]

Nobody disputes the claims of Whitehead and Russell, that their rules of inference for "Peanese" (the symbolic language of Peano), are (i) infallible subject to restrictions stated in English in their text, and (ii) sufficient for much of elementary mathematics. However, the actual mathematical coverage of *Principia* (in nearly

2000 pages!) is far less than Peano's, and it cannot be said that their symbolic methods used "increase the power of reason;" I think they *decrease* it, probably for psychological reasons.[22]

9. Hilbert and Gödel, 1918–31. Because of its capability of replacing special axioms for the different branches of mathematics by theorems (cf. *Principia Mathematica*, Preface, first paragraph), Hilbert said in 1918 that "Russell's Axiomatization of Logic is the crowning achievement of axiomatics."[23] And Hilbert spoke with authority, as the man who had rigorized the axioms of Euclid in his famous *Grundlagen der Geometrie* only 20 years before. I quote from the introduction to this book:

"Geometry—like number theory—requires for its deductive (*folgerichtige*) construction only a few basic theorems (*Grundsätze*). These theorems are called *axioms*,[24] and their connected development has had numerous treatments since Euclid.... The following book is a new attempt to develop the simplest possible complete axiom system for geometry ... so as to clarify the significance of the different groups of axioms and the consequences of the individual axioms."

In much the same spirit of *axiomatic analysis*, Hilbert and his collaborators, especially his co-authors W. Ackermann and P. Bernays,[25] made after 1918 major efforts to prove *deductively* (by metamathematical arguments) the adequacy of the axioms of Whitehead and Russell (the evidence of *Principia Mathematica* was empirical). They focused attention on two main questions:

(i) Are these axioms *contradiction-free*, i.e., using them, is it impossible to prove both p and its contradiction $\sim p$?

(ii) Can one test the truth or falsity of any given proposition (e.g., of arithmetic) in a finite number of steps?

Hilbert may have been attracted to these questions partly because he had established an analog of the first in his *Grundlagen der Geometrie*, by using Cartesian geometry as a model for Euclidean plane geometry, and of the second for polynomial ideal bases by general transfinite arguments using the "ascending chain condition."

Question (i) was given a positive answer by Ackermann (who had earlier proved the redundancy of one of the Whitehead-Russell axioms) and von Neumann in 1927, *under suitable restrictions*. These restrictions, which are quite technical,[26] seemed quite harmless at first sight, and led to a feeling of optimism about Hilbert's program in the years 1927–1930.

Question (ii), the *Entscheidungsproblem* or Decidability problem, was however given a negative answer, even for arithmetic propositions, by Gödel in 1931. By an ingenious use of metamathematical reasoning, ultimately based on Cantor's diagonal construction, he inferred from this undecidability the *incompleteness* of the Whitehead-Russell-Hilbert system in the following sense. Assuming as true the additional *consistency axiom*, that "false formulas are unprovable," one can prove a number-theoretic formula which would not be provable without it. It is a corollary that one

cannot prove that Hilbert's axioms are contradiction-free, so that in particular Question (i) is undecidable.

Thus Gödel's paper shattered Hilbert's high hopes. To quote Hermann Weyl[27]: "Gödel enumerated the symbols, formulas, and sequences of formulas in Hilbert's formalism in a certain way, and thus transformed the assertion of consistency into an arithmetic proposition. He could show that this proposition can neither be proved nor disproved within the formalism. This can mean only two things: either the reasoning by which a proof of consistency is given must contain some argument that has no formal counterpart within the system, i.e., we have not succedeed in completely formalizing the procedure of mathematical induction; or hope for a strictly 'finitistic' proof of consistency must be given up altogether. When Gentzen (1936) finally succeeded in proving the consistency of arithmetic he trespassed those limits indeed by claiming as evident a type of reasoning that penetrates into Cantor's 'second class of ordinal numbers'."

Gödel's result ended abruptly a half-century of optimism about symbolic logic, at least as formalized by Peano, Whitehead, and Russell. It showed that their formalizations were incapable of resolving the paradoxes and ambiguities of Cantor's theory of infinite sets.[28]

THE REIGN OF MODERN ALGEBRA, 1930–1970.

10. The rise of "modern" algebra. Just before Gödel shattered the high hopes of symbolic logicians for formalizing all mathematics in terms of "Peanese", van der Waerden's *Moderne Algebra* (1930–31) precipitated a new revolution. The goal of this brilliantly written book is clearly stated in its preface.

"The 'abstract', 'formal' or 'axiomatic' direction, which has given to algebra renewed momentum,[29] has above all led to a series of new concepts in *group* theory, *field* theory, *valuation* theory, and the theory of *hypercomplex numbers*, to insight into new connections and to far-reaching results. The main aim of this book is to introduce the reader into this new world of concepts."

As I have indicated, both the axiomatic approach and much of the content of "modern" algebra dates back to before 1914. However, even in 1929, its concepts and methods were still considered to have marginal interest as compared with those of analysis in most universities, including Harvard. By exhibiting their mathematical and philosophical unity, and by showing their power as developed by Emmy Noether and her other students (most notably E. Artin, R. Brauer, and H. Hasse), van der Waerden made "modern algebra" suddenly seem central in mathematics. It is not too much to say that the freshness and enthusiasm of his exposition electrified the mathematical world—especially mathematicians under 30 like myself.

In particular, it made classical *real and complex algebra* seem passé, or at least a part of analysis and not of "algebra" in the true sense. This view is exemplified in

Moderne Algebra, where the real and complex fields are not even *defined* until after Galois theory has been presented, and the existence and uniqueness of a smallest algebraically closed extension of *any* field (Steinitz, 1910) are proved purely algebraically (by transfinite induction). What a contrast with the texts of Weber, Serret, and Perron!

11. Lattice theory. This new attitude was a major stimulus in the rebirth of lattice theory, which had lain dormant since the pioneer papers of Dedekind. In 1933, I wrote that lattice theory provided "a point of vantage from which to attack combinatorial problems in ··· abstract algebra."[30] And by 1938, enough progress had been made in applying it to logic, algebra, geometry, probability, measure and integration theory, and functional analysis to cause the American Mathematical Society to hold a symposium on the then very fresh subject.[31]

12. College algebra. The displacement of classical algebra by modern algebra took time. Thus it was not until after World War II that modern algebra became popular at the college level in our country—a popularity due partly to the *Survey of Modern Algebra* which Mac Lane and I had published in 1941. Actually, our approach seems quite conservative today! Thus, unlike van der Waerden, we presented the essentials of the theory of equations before defining groups, and the theory of real and complex matrices (including the Principal Axis Theorem for symmetric and Hermitian matrices) with geometric applications before Galois theory. We also included Boolean algebra, thinking it essential for students to understand the algebra of sets and logic; I shall return to this later.

13. Bourbaki's influence. Abstract mathematics, as reformulated by N. Bourbaki[32] in his *Éléments de Mathématique*, was popularized in French universities not long after. This many-volume treatise, mostly written in the decade 1945–55, attempts to develop all of (pure) mathematics systematically from the notions of *set* and *function*: it presents the content of mathematics as concerned with abstractly conceived relational *structures over sets* and mappings (especially *morphisms*) between them; cf. Book 1, Ch. 4.

Algebraic structures are treated in this spirit in Book 2, as defined by sets of *elements* with (internal or external) finitary *operations*. The reader is then led authoritatively and surely through a carefully polished and systematic sequence of definitions, examples, and theorems about groups, rings, fields, and most of the other kinds of systems I have mentioned. Other branches of mathematics are treated in much the same style in later books. The net effect is to make mathematics appear as a *polished monolith, built purely deductively from the notions of set and function.*

14. The flowering of abstract algebra. The enthusiasm generated by van der Waerden's book, reinforced in the ways that I have described, has given rise to an unprecedented flowering of all aspects of abstract algebra over the past 40 years. In

particular, the theories of *groups*, *rings* and *fields* (to which the bulk of *Moderne Algebra* was devoted) have achieved new levels of depth and sophistication, of which perhaps the most dramatic example is the result that *every finite group of odd order is solvable*. This result, proved by Thompson and Feit in over 200 pages of very technical reasoning, had long been conjectured—but to prove it would have seemed hopeless to most mathematicians in 1930.

The last 40 years have also seen the theories of Lie, Jordan, and multilinear algebras mature to a point that makes what was known in 1930 seem amateurish if not naive. The same is true of lattice theory, semigroup and quasigroup theory, category theory, and homological and combinatorial algebra, all of which were either unknown or nearly so in 1930. Finally, algebraic geometry has become rigorized as a new branch of axiomatic algebra, based securely on deep results about commutative rings and their ideals and valuations.[33]

15. Wider repercussions. The tidal wave generated by enthusiasm about abstract algebra had wider repercussions. Thus to young men in 1930, like myself, van der Waerden's book made *classical analysis* stemming from the calculus ("*analyse infinitésimale*"), which had dominated mathematics for over two centuries, suddenly seem old and tired. Indeed, the abstract approach adopted by van der Waerden for algebra soon became fashionable in functional analysis and topology. The idea that all mathematics could be viewed as topological algebra gained a strong impetus from the solution of Hilbert's Fifth Problem, which showed that the hypothesis of differentiability could be replaced by mere continuity in the theory of Lie groups: any locally Euclidean continuous group is isomorphic to an analytic Lie group [**22**, p. 184]. Even research on partial differential equations, the traditional stronghold of the applied mathematician, has increasingly centered around the quest for new abstract concepts permitting one to prove extremely general existence and uniqueness theorems.

Partly because of such shifts in emphasis, by 1960 most younger mathematicians had come to believe that all mathematics should be developed axiomatically from the notions of set and function, and this approach had come to seem no longer modern but classical! By 1959, van der Waerden had changed his title from "*Moderne Algebra*" to "*Algebra.*" And in the 1960's, Mac Lane and I wrote another "Algebra" which went further in the direction of abstraction, by organizing much of pure algebra around the central concepts of morphism, category, and "universality." The "universal" approach to algebra, which I had initiated in the 1930's and 1940's stressing the role of lattices, was developed much further in two important books by Cohn and Grätzer. In a parallel development, Lawvere (1965) proposed "The category of categories as a foundation for mathematics," beginning with the statement[34]:

> In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics

which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular, in which classes and membership in classes do not play any role. Here by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved. A foundation of the sort we have in mind would seemingly be much more natural and readily-usable than the classical one when developing such subjects as algebraic topology, functional analysis, model theory of general algebraic systems, etc.

16. The "new mathematics" of 1960. In the post-Sputnik era of the early and middle 1960's, enthusiasm went even further. Especially in the United States, a vogue developed for exposing school children to formal concepts of set, function and axiom often only half-appreciated by their teachers! Its proponents encouraged the spread of the myth that these constituted a "New Mathematics," unknown fifty years earlier. One ostensible aim of this vogue was to indoctrinate young people so that they could fill a supposed shortage of mathematical teachers and research workers. This seemed highly desirable at a time when our postwar "baby bulge" and prosperity was quadrupling of the demand for college teachers of mathematics, while an unquestioning faith in the value of basic science was increasing the support for research in pure mathematics at a rate of 10–15 per cent annually. But as of 1972, it all seems strangely out-of-date!

To summarize, algebra developed harmoniously during the years 1930–60, with its main stream flowing smoothly, swiftly, and finally triumphantly in the channels I have described. Some measure of its triumph may be found in the fact that, whereas three of the first four Fields medals were awarded in Analysis (in 1936 and 1950), three of the four awarded in 1970 were in Algebra.

However, in the last 5–10 years, powerful new currents have become apparent. Some of these have arisen as countercurrents to extremism; thus René Thom has recently written a thought-provoking article entitled *'Modern' Mathematics: An Educational and Philosophical Error*,[35] in which he urges that geometry should replace algebra because "any question in algebra is either trivial or impossible to solve. By contrast, the classic problems of geometry present a wide variety of challenges."

However, I do not wish to dwell on the exaggerations of a decade which most of us recall with nostalgia. Extreme abstraction in research circles, attempts to inculcate premature sophistication in children, and uncritical expansionism in basic physical science have provoked reactions which by now threaten to go too far in the opposite direction.

Instead, I wish to describe four *positive* current trends in algebra which, in my opinion, hold great promise for the future.

FOUR COMPUTER-INFLUENCED CURRENT TRENDS

17. The new numerical algebra. Already in the 1940's a new revolution was brewing, whose ultimate implications for mathematics are unpredictable. Namely,

the construction of efficient *high-speed digital computers* was making it feasible to solve mathematical problems whose effective solution would have previously been prohibitively costly and time-consuming. To many mathematicians, including myself, it had become evident by 1950 that the resulting *revolution in applied mathematics* would open up challenging new areas for basic research. In particular, since digital computers can only represent real numbers to a *finite* number of significant digits, and can only represent values of real functions at a *finite* number of points (approximate "nodal values" at "mesh-points"), their use in solving differential equations (e.g., from physics or engineering) requires a very careful *numerical analysis* of *roundoff* and *truncation* errors.[36]

Thus, to actually *solve* a system of differential equations (to a desired approximation), one usually first replaces it by an approximating system of *algebraic equations* (obtained perhaps by finite difference or finite element methods), whose unknowns typically represent nodal values at mesh-points, which is then solved (also approximately) on a digital computer. I shall say nothing about this first step of *discretization* here, because the theorems in numerical analysis and approximation theory required to justify it belong to classical analysis and not to algebra. Suffice it to say that it often leads to very large matrices and associated systems of simultaneous linear equations, which may involve 50,000 or more unknowns! The main problem is to solve these efficiently.

These matrices typically have many special properties, which must be exploited to achieve efficiency. They are usually very *sparse* (have mostly zero entries), and often symmetric, or symmetrizable by permutations or linear transformations. Their diagonal elements may be "dominant" (i.e., at least as great as the sum of the absolute values of the other entries), and they may have positive diagonal and negative off-diagonal entries. Matrices having all of the above properties are essentially what are called *Stieltjes matrices*; they arise naturally in *network flow problems*.

One usually wants to either: (i) *solve* the linear system (written symbolically $Ax = b$), or (ii) determine *eigenvalues* of A (the former are of course the roots of $|A - \lambda I| = 0$). As regards (i), most mathematicians imagined in 1940 that large linear systems should be solved (if at all!) by *Gaussian elimination*, and that the rest was sheer drudgery. A few eminent analysts (including Gauss, Jacobi, and von Mises) had appreciated the value of *iterative* methods (also used by Gauss) and had studied their rates of convergence, but these methods were (and still are!) totally ignored in textbooks on "linear algebra." Similar remarks apply to eigenvalue problems, where the experience of most mathematicians was limited to 3×3 (if not to 2×2) matrices $A = \| a_{ij} \|$, whose eigenvalues they might have found using textbook formulas to solve the cubic characteristic equation

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \beta\lambda - A = 0,$$

where

$$\beta = a_{22}a_{33} + a_{33}a_{11} + a_{11}a_{22} - a_{23}a_{32} - a_{31}a_{13} - a_{12}a_{21}.$$

In practice, such textbook methods are extremely inaccurate and inefficient for most large matrices[37], and they were replaced in the 1950's by new algorithms, whose invention and analysis created a major new area of "classical" algebra: the *new numerical algebra.* Excellent surveys of what is now known about this area are contained in authoritative books by Varga [**17**], Wilkinson [**19**], and Young [**20**]; every forward-looking young algebraist should at least be cognizant of their contents!

18. Sparse matrices. The past five years have also seen substantial improvements (over Gauss) in *elimination* techniques for solving large systems with sparse coefficient-matrices. In particular, these have drawn on graph theory for ideas; see [**15**] for a cross-section of current work.

There are many other interesting new areas of research in (real and complex) numerical algebra. I shall just mention three of the most important; references to activity in them may be found in many review journals:

(a) Finding the roots of polynomial equations of degrees up to 100.
(b) "Unconstrained" minimization of functions of many variables.
(c) Linear programming and other techniques for finding minima of functions subjected to "constraints" by equations and inequalities.

Actually these "new" areas also originated in the 1940's, if not earlier. Thus by 1947, linear programming was defined, and the "simplex method" of solving its problems invented by George Dantzig; see p. 20 of G. Hadley's *Linear Programming* (Addison-Wesley, 1962). Moreover, its fundamental techniques were made accessible at the college freshman level by Kemeny, Snell, and Thompson 10 years later, in their popular *Introduction to Finite Mathematics* (Prentice-Hall, 1957).

19. Integer arithmetic. In programming languages for computers, a basic distinction is made between *exact* "integer arithmetic" and *approximate* "real arithmetic." I have omitted the problems of "integer programming" and of solving Diophantine equations on computer in the above discussion, because they involve integer and not real and complex numerical algebra. Nevertheless, activity in these fields represents another strong trend in contemporary numerical algebra.

20. Theory of automata. Although many mathematicians think of high-speed computers as simply "number-crunchers" or supersliderules whose primary mathematical role is to carry out elaborate numerical computations, and although "arithmetic units" may be the most highly organized special pieces of computer "hardware," computers are actually much more versatile. Large general purpose computers are designed to be *universal* instruments, capable of expediting all kinds of "mental" tasks. Much as the Industrial Revolution was made possible by machines which could perform all kinds of "physical" tasks more cheaply and efficiently than human beings, the Computer Revolution is aimed at doing the same with mental tasks. This prospect makes the study of computers especially fascinating. From a mathematical standpoint, partly because general purpose computers are *digital* assemblies of a

finite set of components, their study is based on a new, *purely algebraic* concept which I shall now define axiomatically.

DEFINITION. A *finite state machine* (or "automaton") M consists of a collectiom A of "input symbols," a collection S of "states," and a collection Z of "output symbols," related by two operations $\nu\colon A \times S \to S$ and $\zeta\colon S \times Z \to Z$. The operation ν assigns to each "input symbol" $a \in A$ and "prior state" $s \in S$ a "new state" $\nu(a, s) \in S$; the operation ζ assigns to a and s a "printout" $\zeta(a, s) \in Z$. More concretely, such a finite state machine M can be thought of as evolving from a specified *starting state* s_0, *recursively* by $s_k = \nu(s_{k-1}, a_k)$, and as *printing out* $z_k = \zeta(s_{k-1}, a_k)$ for $k = 1, \cdots, n$ in succession. In this way, it converts strings of input symbols or *programs* $a_1, a_2, \cdots, a_n$ into printouts $z_1, z_2, \cdots, z_n$.

Abstractly, a finite state machine is clearly just a new kind of algebraic system $M = [A, S, Z; \nu, \zeta]$. If one simplifies M by ignoring Z and ζ (this is called a "forgetful functor" in category theory), the simplified M just describes the *action of a free semigroup* (the set A^* of all possible input "programs") *on a set* (the set S of states). The resulting theory of state machines without output fits nicely into axiomatic (or "modern") algebra and, as has recently been shown,[39] so-called "universal algebra" can be applied to it.

21. Turing machines. Quite similar to finite state machines, but a little more complicated, are the "Turing machines" invented by the logician Turing in 1936, before high-speed general purpose digital computers existed. Turing proved that they could indeed carry out most processes of mathematical "thought." Thus they are capable of printing out the binary or decimal expansion of any "definable" (alias "computable") real number, such as e, π, or the kth zero of the Bessel function $J_n(x)$, and they can "deduce all the provable formulas of the *restricted* Hilbert functional calculus," giving all true theorems and no false ones.

Some two decades after Turing showed that his machines could, *in principle*, carry out the kind of mechanical theorem-proving dreamed of by Leibniz, Whitehead and Russell, and Hilbert, Hao Wang did this *in practice*. Namely, he wrote a special program which produced "proofs" in minutes for all the 350 theorems in the predicate calculus with equality that were actually stated in Whitehead and Russell's *Principia Mathematica*![40]

22. Computational complexity; optimization. A third and very strong trend in algebra, and indeed in mathematics generally, is a concern with *computational complexity* and with *optimization*. In all *applications* of algebra, of course, the efficiency of symbol-manipulation is a prime consideration, but for many years it was taboo to discuss it in research journals devoted to pure mathematics.

This snobbish taboo against discussing efficiency obscured some very important basic facts. Thus, in the area of mathematical logic, the scholarly books by Whitehead and Russell and the Hilbert school did *not* seriously try to improve the

efficiency of formal deductive schemes, whereas Leibniz and Peano were really trying to (and did; especially Leibniz!) develop symbolic techniques for making mathematical reasoning more efficient and, therefore, more powerful. This difference becomes painfully obvious if one compares the number of symbols required by Whitehead and Russell to derive the basic formal properties of sets and relations, with the number of words needed by mathematicians to get equally far. So far, it is only in the area of the propositional calculus of logic itself, and by using a powerful computer, that mechanical theorem-proving has been realized on a substantial scale (by Hao Wang, see §21).

Having finally recognized the importance of efficiency, mathematical logicians have begun to analyze the "computational complexity" of applying general definitions to particular cases. Their analysis has already borne fruit in the development of shorter procedures for multiplying numbers and matrices.

Concern with computational complexity in algebra has as its ultimate goal, of course, the *optimization* of symbolic methods. In turn, the question of optimization has already suggested a number of basic problems whose solution should be a continuing challenge, stimulating coming generations of *pure* algebraists. Two of these are, respectively: (i) the "shortest form" problem of Boolean algebra, and (ii) the "most efficient coding" problem of information theory.

Other fascinating optimization problems, concerning which surprising discoveries have recently been made, are: (iii) what is the least number of operations on digits required to multiply two n-digit integers? (iv) what is the smallest number of arithmetic operations required to multiply together two $n \times n$ matrices? (v) how can one solve n simultaneous linear equations in n unknowns with the fewest additions, subtractions, multiplications and divisions? I regret that I do not have time to discuss these problems here, and must refer you to the literature ([**21**] and [**11**, vol. 2, pp. 258–78]).

23. Combinatorial algebra. A fourth current trend in algebra is towards emphasis on *combinatorial* ideas,[41] and especially on those involving *graphs* or *networks*. This trend is surely due to an intuitive recognition of the fact that digital computers and the deductive procedures of mathematics have a structure whose analysis requires combinatorial methods. As Hermann Weyl wrote in 1949: "The network of nerves joining the brain with the sense organs is a subject that by its very nature invites combinatorial investigation. Modern computing machines translate our insight into the combinatorial structure of mathematics into practice by mechanical and electronic devices."[42]

From burgeoning elementary courses in "Discrete Mathematics" which are intended to *precede* courses in axiomatic algebra,[43] probability and statistics, to the ambitious 7-volume treatise [**11**] on *The Art of Computer Programming* being written by Donald Knuth, the new emphasis is the same: permutations, combinations,

partitions, generating functions, trees, sorting, searching, recurrences, and difference equations, block designs, and so on. Even a casual reading of the books I have cited makes it very clear that the 200 year reign of the Calculus and Analysis has ended—and that they will continue to be displaced in our colleges by courses in Algebra in the broadest sense of discrete mathematics and the *science* (no longer just art!) of symbol manipulation.

In a sense, this trend continues the revolution begun by van der Waerden, but there has been a major change. No longer do axioms and deductive systems, patterned after Euclid's *Elements*, seem so fundamental. Neither do groups or rings, with their subgroups, subrings and morphisms. Their place is taken by various *relational structures* (including partial orderings and "complexes" in the sense of combinatorial topology) which are far less amenable to the general algebraic techniques which played such a central role in the "modern algebra" of 1930–1960.

Instead, the kinds of algebraic structures (as contrasted with "relational" structures) which are most relevant to digital computers and combinatorics are loops, monoids and lattices (or groupoids, semigroups and semilattices), which were largely ignored by most algebraists in 1930–1960. Loosely speaking, much as *groups* are related to *symmetries*, so *loops* are related to *designs* (or "patterns"), *monoids* to *actions* (e.g., of input instructions on the states of an automaton), and *lattices* to *structure.*

In particular, Rota[44] and his associates have shown that lattice theory provides a point of vantage from which to attack combinatorial problems in general, and not just those of algebra as I had stated in 1933 (see §7). Going even further, N. S. Mendelsohn has very recently applied concepts of universal algebra to generate combinatorial designs and vice-versa [**23**, pp. 123–32].

One naturally wonders where all these new trends will lead to. I am myself sure of only one thing: that they will *not* make the classical "modern algebra" expounded in van der Waerden obsolete, any more than this made real and complex algebra or the calculus obsolete. As Knuth emphasizes ([**11**, vol. 1, p. 1]; see also [**1**]) the word *algorithm* (or "algorism") which is so central to the mathematics of computation is just a corruption of the name Al-Khwarizmi, the originator of the word "algebra."

Indeed, the four current trends in algebra which I have been describing were merely *stimulated* by the consideration of digital computers, in much the same way that the calculus and analysis were stimulated by thinking about geometry, mechanics and mathematical physics. They are simply opening up new areas of mathematics for future generations to study, with an ever increasing variety and richness of interrelations and applications, in which old and new ideas will mingle and be reshaped. Within a few decades, new concepts and trends may well emerge from this mingling and reshaping. Certainly, this kind of continuing evolution is the only thing that can keep algebra perennially a fresh and exciting subject!

Notes

[1] See footnote on page 761.

[2] For this and other facts, I am indebted to Professor David Pingree of Brown University; Thomas Hawkins, Gian-Carlo Rota, Gerald Sachs, and John Tate made other very helpful comments.

[3] A notable exception is provided by the binomial theorem, discovered by Pascal in 1653. For readable accounts of the facts summarized in this section, see Rouse Ball [1] and E. T. Bell [3].

[4] Their expositions were very obscure; see G. Birkhoff, *Isis* 3 (1973), 260–7. That of Galois was clarified by Betti in 1852.

[5] We here follow the usual custom of letting Z (for the German "*Zahl*" meaning integer) stand for the set $\{0, \pm 1, \pm 2, \cdots\}$. Gauss attributed the consideration of integers mod n ("modular numbers") to Legendre.

[6] For penetrating historical surveys of linear and non-commutative algebra, see N. Bourbaki, [6, pp. 78–91 and 120–28]. For a readable summary of Cayley's contribution, see pp. 102–15 of E. T. Bell [2].

[7] Gergonne's Annales 5 (1814–15), p. 93; for Hamilton's ideas, see his *Mathematical Papers*, vol. III, Cambridge Univ. Press, 1967. Leibniz and Cramer had very fragmentary ideas about determinants; see [1, p. 375] and D. J. Struik, *A Source Book in Mathematics*, Harvard University Press, 1969, p. 180.

[8] R. Woodhouse, Phil. Trans. 91 (1801), 89–119; G. Peacock, Reps. Brit. Assn. Adv. Sci. 3 (1834), 185–32 and *Algebra*, 2 vols., 1845; A. de Morgan, Trans. Camb. Phil. Soc., 7 (1839) 173–87 and 287–300; G. Boole, Cambridge and Dublin Math. J., 3 (1848) 183–98.

[9] F. Klein, *Entwicklung der Mathematik im 19ten Jahrhundert*, vol. 1, p. 175, characterized this as "almost unreadable."

[10] In his supplements to Dirichlet's *Vorlesungen über Zahlentheorie*, 1863, 1871.

[11] Benjamin Peirce, *Linear Associative Algebra*, Boston, 1870; see also Amer. J. Math., 3 (1880) 15–57, and 4 (1881) 97–229 (reprinted from Proc. Am. Acad. Boston, 1875).

[12] See his *Collected Papers*. vol. 2, Cremonese, Rome, 1958, p. 134. In the Amer. Math. Society *Semicentennial Addresses*, vol. 2, p. 15, Bell attributed the postulational approach to Peano! Peano was also the first to *number* his theorems.

[13] Volumes 3–6 of the Transactions of the (then young) American Mathematical Society (1902–5) contain a dozen articles on postulate systems by the men named above.

[14] Eulcid's *Elements*, which included "axioms" for magnitudes (algebra) as well as "postulates" for geometry, were written in Alexandria, Egypt, around 300 B. C.; see Ball [1].

[15] For a historical discussion of ideal theory and Dedekind's work on algebraic numbers, see [3, Ch. 10].

[16] Op. cit. supra, pp. 216–29. The same result was proved independently by Frobenius, op. cit. infra.

[17] G. Frobenius, Crelle, 84 (1878) 1–63, and Berlin Sitzb. (1903) 504–37 and 634–5. Wedderburn's *Lectures on Matrices*, Amer. Math. Soc., 1934, contains a complete bibliography to 1933.

[17a] See Thomas Hawkins, Archive for History of Exact Sciences, 8 (1972) 243–87.

[18] A related symbolic style of writing was used by E. H. Moore in his *Introduction to a Form of General Analysis*, New Haven Colloquium, Yale Univ. Press, 1910.

[19] See F. Cajori, "Past struggles between symbolists and rhetoricians. . .", Proc. Int. Math. Congress Toronto (1924), vol. 2, pp. 937–41.

[20] First published in 1889 (*Arithmetices principia nova methodo exposita*).

[21] The fact that this was so had been airily asserted a decade earlier by Russell in his witty *Principles of Mathematics*, of which *Principia Mathematica* was originally intended to be comprised in a second volume!

[22] See G. Birkhoff, "Mathematics and Psychology," SIAM Review, 11 (1969) 429–69.

[23] *Werke*, vol. 3, p. 153; Math. Annalen 78, 405–15.

[24] Hilbert is here slurring over Euclid's distinction between "axioms" (for magnitudes in general) and "postulates" (for geometrical entities).

[25] Of the books **[10]** and *Grundlagen der Mathematik* (2 vols., 1939), respectively.

[26] See S. C. Kleene, *Introduction to Metamathematics*, Van Nostrand, 1932, pp. 204–5.

[27] This MONTHLY, 53 (1946) 1–18. Gödel's original paper was published in the Monats. Math. Phys., 38 (1931) 173–98.

[28] Careful historical reviews of the question touched on here may be found in N. Bourbaki, **[6,** Ch. 1], and (by P. Bernays) in Hilbert's *Werke*, vol. 3, pp.. 196–217; this volume also contains Hilbert's papers on logic.

[29] In German, "der die Algebra ihren erneuten Aufschwung verdankt."

[30] Proc. Camb. Phil. Soc., 29 (1933) 441.

[31] Bull. Amer. Math. Soc., 44 (1938) 793–827.

[32] A pen-name assumed in 1937 by a group of then young French mathematicians, who wished to overthrow the domination of French mathematics by classical analysts. See this MONTHLY, 57 (1950) 221–32 for authentic statement of Bourbaki's opinions, including the view that the axiomatic method is "a *standardization* of mathematical technique," and that the principal mathematical structures are those of a group, of order, and of a topological space.

[33] For example, anyone doing serious research on algebraic "geometry" today is expected to consider the two-volume treatise on *Commutative Rings* by O. Zariski and P. Samuel as standard *preliminary* material, but not to know Newton's classification of real cubic curves!

[34] F. William Lawvere, "The category of categories as a foundation for mathematics," *Proc. Conf. Categorical Algebra*, La Jolla, 1965 (S. Eilenberg *et al*, eds.), Springer, 1966.

[35] *American Scientist*, Nov. - Dec., 1971.

[36] Mathematicians habituated to exclusively deductive reasoning should realize that, in practice, error analysis relies very heavily on empirical evidence as well as on theoretical principles.

[37] Though not as nearly inefficient as Cramer's Rule, which is still often the only prescription given to students!

[38] See Marvin Minsky, *Computation: Finite and Infinite Machines*, Prentice-Hall, 1967.

[39] G. Birkhoff and J. D. Lipson, "Heterogeneous Algebras," J. Comb. Analysis, 2 (1969).

[40] H. Wang, IBM J. Res. Develop., 4 (1960) 2–22. For the general question of the computer as a "brain," see the reference of note 22.

[41] Wallis, Tchirnhaus, and Leibniz all recognized before 1700 that combinatorics belonged to algebra. See **[13,** p. 14] and **[21,** p. 2].

[42] E. F. Beckenbach (editor), *Applied Combinatorial Mathematics*, Wiley, 1964, p. 537.

[43] As currently recommended by the CUPM Panel on the Impact of Computing on Mathematics Courses. On an intermediate level, see C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, 1968; on a more advanced level, see M. Hall, *Combinatorial Theory*, Ginn, 1967.

[44] "On the foundations of combinatorial theory," J. für Wahrsch., 2 (1966) 340–68; *Combinatorial geometries* (preliminary edition), M.I.T. Press, 1970; and refs. given there.

References

1. W. W. Rouse Ball, A Short History of Mathematics, 3rd ed., Macmillan, New York, 1901.

2. Eric T. Bell, Mathematics: Queen and Servant of Sciences, McGraw-Hill, New York, 1951.

3. ———, The Development of Mathematics, McGraw-Hill, New York, 1940.

4. Garrett Birkhoff and Thomas C. Bartee, Modern Applied Algebra, McGraw-Hill, New York, 1970.

5. Garrett Birkhoff and Saunders Mac Lane, A Survey of Modern Algebra, Macmillan, New York, 1941.

6. Nicolas Bourbaki, Éléments d'Histoire des Mathématiques, Hermann, Paris, 1960.
7. Florian Cajori, A History of Mathematical Notations, 2 vols., Open Court, Chicago, 1928–9.
8. George Grätzer, Universal Algebra, Van Nostrand, Princeton, N. J., 1968.
9. David Hilbert, Grundlagen der Geometrie, 1899; 2nd. ed., 1901. Authorized translation by E. J. Townsend, Open Court, Chicago, 1902, 1910.
10. David Hilbert and W. Ackermann, Grundzüge der theoretische Logik, 4th ed., 1949.
11. Donald Knuth, Algorithms, 7 projected volumes, Addison-Wesley, Reading, Mass., 1969.
12. S. Mac Lane and G. Birkhoff, Algebra, Macmillan, New York, 1967.
13. Uta Merzbach, "... Development of Modern Algebraic Structures from Leibniz to Dedekind," Ph. D. Thesis, Harvard, 1965.
14. Giuseppe Peano, Formulario Matematico, 4th ed., Torino, 1908.
15. Donald Rose and Ralph Willoughby (eds.), Sparse Matrices and their Applications, Plenum Press, New York, 1971.
16. B. L. van der Waerden, Moderne Algebra, 2 vols., Springer, New York, 1930–31.
17. Richard S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962.
18. Alfred N. Whitehead and Bertrand Russell, Principia Mathematica, 3 vols., Cambridge Univ. Press, 1911.
19. James Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1966.
20. David M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.
21. Garrett Birkhoff and Marshall Hall (eds.), Computers in Algebra and Number Theory, SIAM-AMS Proceedings, vol. IV, Amer. Math. Society, 1971.
22. Deane Montgomery and Leo Zippin, Topological Transformation Groups, Wiley-Interscience, New York, 1955.
23. W. Tutte (ed.), Recent Progress in Combinatorics, Academic Press, New York, 1969.

MATHEMATICAL NOTES

EDITED BY ROBERT GILMER

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

EXISTENCE OF FOUR CONCURRENT NORMALS TO A SMOOTH CLOSED HYPERSURFACE OF E^n

BERND WEGNER, Technische Universität Berlin

A four-normals-point of a smooth closed hypersurface of E^n (n-dimensional euclidean space) is a point where at least four normals of the hypersurface intersect. There are several examples of four-normals-points of a closed curve in the plane (compare Chakerian and Stein [2], Deo and Klamkin [3], Guggenheimer [4]). The purpose of this note is to give many examples of four-normals-points of a smooth closed hypersurface F of E^n. We shall prove that every neighborhood of a focal point of F contains a four-normals-point of F if F is an immersed $(n-1)$-sphere.