# PARTIAL ORDERINGS AND MOORE-SMITH LIMITS*

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**1. Introduction.** Everybody knows that the concept of limit is fundamental in analysis. But somewhere about the time he reaches advanced calculus, a student may perhaps begin to wonder how many disguises this concept of limit can assume. Apart from the fact that the dependent variable which is doing the converging may be a real number or a complex number or a vector or a function or what not else, the independent variable may also take any one of a multitude of forms. Thus if the dependent variable is real, we still have to consider convergence of sequences of reals, of multiple sequences, of functions of a real variable $x$ as $x$ tends to some $x_0$ or as $x$ tends to infinity, of functions of several real variables, and so on. The saving feature is of course that all these assorted definitions have strong resemblances. Nevertheless, the student may be forgiven if he wishes that somebody would put all these different but related ideas into fewer packages, so that it wouldn't be necessary to do almost the same thing over and over again for the slightly different kinds of limit. This may occur to him even if he has never heard of E. H. Moore's dictum, that the existence of strong similarities in the central part of different theories indicates the existence of a more general theory of which these different theories are all special cases. But in such a situation as this there is something to be considered beside generality. What we really want is a treatment of the subject which is not only unified, but is also elegant and easily understood. As a matter of fact, such a theory has been in existence for twenty-nine years. The treatment of limits† devised by E. H. Moore and his then student H. L. Smith appeals to me as one which can readily fit into a course in the theory of functions of a real variable or a course in advanced calculus, not only strengthening the content but also providing a continuous thread tying the subject matter together. And besides this, their treatment has another virtue which is important to a beginner and by no means to be despised by those who are no longer beginners; it follows closely the lines of the simplest of all limit theories, that is the theory of convergent sequences, follows them indeed so closely that many of the proofs can be taken over with only trivial notational changes. Thus the theorems for the more complicated limiting processes are obtained along with the simpler ones, and at a cost in effort little greater than that demanded by the simplest of all.

Let us then look over some of the definitions of limit that an undergraduate might be expected to know, and try to separate the essential parts shared by the different examples. From one point of view, we could classify all the defini-

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† Moore, E. H. and Smith, H. L., A general theory of limits. Amer. J. Math., vol. 44, 1922, p. 102. Here we should also mention the very inclusive theory of limits based on filters, devised by H. Cartan, Theorie des filtres, C. R. Acad. Sci. Paris, vol. 205, 1937, pp. 595–598, Filtres et ultrafiltres, ibid., pp. 777–779. This same theory was also expounded by H. L. Smith, apparently independently (A general theory of limits, Nat. Math. Mag., vol. 12, 1938, pp. 371–379).

tions into two principal varieties. The first type is exemplified by the limit of a function $f(x)$, defined for all real $x$, as $x$ approaches $x_0$. Roughly stated, the limit of $f$ is $l$ if $f(x)$ is near $l$ whenever $x$ is near $x_0$. The other type is exemplified by the limit of a sequence. The limit of $f_n$ $(n=1, 2, \cdots)$ as "$n$ tends to infinity" is $l$ if $f_n$ is near $l$ for all $n$ after a certain $n_0$. If we would generalize the first type, we should study the idea "$x$ is near $x_0$" and seek to give it a meaning both precise and general. Thus we are led to study neighborhoods, and eventually arrive at the idea of a topological space. In this day it would be utterly superfluous to stress the importance of the study of topological spaces. But at the moment we are not going in that direction. It is the other type of limit that is to engage our attention.

**2. Directed sets and nets.** What then are the essential elements of the definition of the limit of a sequence? To begin with, the property that $n$ is an integer is quite irrelevant since the same kind of definition applies to the limit of $f(x)$ as $x$ tends to $\infty$ over the whole real number system. But the relation "$n$ is after $n_0$," or in symbols $n>n_0$, is important. To begin with, it is a binary relation on the integers. That is, if we write an integer $n$, and then write the name $>$ of the relation, and then write another integer $m$, we obtain a meaningful sentence "$n>m$"; the subject is "$n$," the verb is "$>$," and the object is "$m$." The sentence may be true or it may be false, but it is not nonsense. This is what we shall mean by a binary relation on a set $A$; $R$ is a binary relation on $A$ if whenever $a$ and $b$ are members of $A$, "$aRb$" is a meaningful sentence, either true or false.

But a relation that could in any reasonable sense be thought of as "after" must have another property. If $a$ is after $b$ and $b$ is after $c$, we surely can reasonably ask that it should also be true that $a$ is after $c$. This is the property called "transitivity." Formally, a binary relation $R$ on $A$ is transitive if whenever $a$, $b$ and $c$ are elements of $A$ such that $aRb$ and $bRc$, then it is true that $aRc$. We shall consider that binary relations $R$ with this property are important enough to be dignified by the name of "partial orderings"; so henceforth a "partial ordering" of $A$ shall be a transitive binary relation on $A$. (But not all mathematicians agree on this meaning, or on any other, for the expression "partial ordering.") Also, we shall make a slight notational change as a crutch to memory; instead of the non-committal name $R$ for the partial ordering, we shall usually use $\succ$, as a reminder of its kinship with $>$. The statement "$a\succ b$" could be read "$a$ follows $b$."

However, we have not yet completed our task of finding all the essential properties used in the definition of a sequence. To show this, we observe that if we define $n\succ m$ to mean that $n-m$ is a positive even integer, $\succ$ is a partial ordering. We could repeat the definition of limit of a sequence with the single change of replacing $>$ by $\succ$, and obtain an intelligible definition. But if we define $f_n$ to be 0 if $n$ is even and 1 if $n$ is odd, we find that $f_n=0$ if $n\succ 2$, so $\lim f_n$ is 0; while $f_n=1$ if $n\succ 3$, so $\lim f_n$ is 1. This is most undesirable. Let us

therefore look back at the proof that a sequence cannot have two different limits. Suppose that $h$ and $k \neq h$ are both limits of $f_n$; let $\epsilon$ be a positive number less than the difference between $h$ and $k$. For all $n$ after a certain $n'$ we have $f_n$ within $\epsilon/2$ of $h$; for all $n$ after a certain $n''$ we have $f_n$ within $\epsilon/2$ of $k$. *Now choose an $n$ which is after both $n'$ and $n''$.* For this $n$, $f_n$ differs by less than $\epsilon/2$ from both $h$ and $k$, which therefore must be within $\epsilon$ of each other. But this contradicts the definition of $\epsilon$, and so the sequence cannot have two different limits. If we try to carry this through with the relation $\succ$ in place of $>$, we find that the trouble occurs at the italicized sentence; if $n'$ is even and $n''$ is odd, there is no $n$ such that $n \succ n'$ and $n \succ n''$. This partial ordering has the peculiarity that if $n \succ m$, then if $m$ is even so is $n$, and if $m$ is odd so is $n$.

Accordingly, we avoid the trouble of non-unique limits by limiting our attention to those partial orderings in which this situation never arises. That is, we assume that our partial orderings have the property that for each pair of members of the set (not necessarily distinct) there is an element of the set which follows both of them. Moore and Smith gave this property a name* which has fallen into disuse, probably because the property has no importance except when combined with transitivity. When a relation $\succ$ partially orders a set $A$ and also has the property just described, it is now customary to say that the relation $\succ$ "directs" the set $A$. To state it in full:

*A set $A$ is directed by a relation $\succ$ if $\succ$ is a binary relation on $A$ with the properties:*

(i) *if $a$, $b$, and $c$ are elements of $A$ such that $a \succ b$ and $b \succ c$, then $a \succ c$;*

(ii) *If $a$ and $b$ are elements of $A$, there exists an element $c$ of $A$ such that $c \succ a$ and $c \succ b$.*

One more bit of terminology is called for. A function which assigns to each positive integer $n$ a real number $f_n$ as functional value has long been known as a sequence of real numbers; more generally, a function which assigns to each $n$ a functional value $f_n$ in a set $M$ is called a sequence of elements of $M$. We wish to have a name for a function which assigns to each element $a$ of a directed set $A$ a real number $f(a)$ as functional value. For this J. L. Kelley† proposes the name "net"‡ of real numbers; more generally, if $f$ is a function which assigns to each element $a$ of a directed set $A$ a functional value $f(a)$ in a set $M$, we shall call the function $f$ a "net" of elements of $M$.

**3. Definition of convergence.** Now we are in a position to attempt to take over the whole theory of convergent sequences, replacing the word "sequence"

* Compositive.

† J. L. Kelley, *Convergence in topology*, Duke Math. J., vol. 17, 1950, p. 277.

‡ Kelley writes me that this was suggested by Norman Steenrod in a conversation between Kelley, Steenrod and Paul Halmos. Kelley's own inclination was to the name "way"; the analogue of a subsequence would then be a "subway"! Since a stream with its tributaries is a good example of a system directed by the relation "downstream from," I incline toward "stream" rather than "net." But "net" has the great advantage of having seen print first.

by "net," the sign $>$ by $\succ$, and the integers by the elements of the directed set $A$. For example, the definition of limit becomes:

DEFINITION. *Let $f(a)$, $a$ in $A$, be a net of real numbers, and let $k$ be a real number. Then* $\lim_{a \text{ in } A, \succ} f(a) = k$ *means that for every positive $\epsilon$ there is an element $a_\epsilon$ of $A$ such that $|f(a)-k| < \epsilon$ whenever $a \succ a_\epsilon$.*

For simplicity of notation we shall omit such parts of the symbolism under lim as can be left out without danger of confusion. For example, if $\succ$ is the only partial order of $A$ that we are considering, we omit mentioning it; if all the functions we are interested in at a given moment are defined on the same directed set $A$, we condense the symbol to $\lim_a f(a)$ or even to $\lim f$. But this is a device familiar to all of us. By such simple devices we take over all the essential parts of the theory of limits. For example, if $f(a)$ and $g(a)$ are both defined and real valued for all $a$ in a directed set $A$, and have the respective finite limits $h$ and $k$, then $f+g$ has the limit $h+k$. As another important theorem, let $f$ be a net of real numbers defined on a directed set $A$ and having the property that $f(a) \geqq f(a')$ whenever $a \succ a'$. (Such nets may be called "monotone non-decreasing," but I think that Garrett Birkhoff's terminology "isotone" is much to be preferred. The corresponding name for nets such that $f(a) \leqq f(a')$ when ever $a \succ a'$ is "antitone," to replace the older expression "monotone non-increasing.") The ordinary proof for sequences carries over to show that if $f$ has this property and also has a finite upper bound, it has a limit, and the limit is the same as the least upper bound.

The proof of the Cauchy criterion for convergence could be taken over too, but we shall sketch one which has the virtue of applying in any metric space by merely replacing absolute differences by distances. Suppose it known that the Cauchy condition is necessary and sufficient for the convergence of sequences. Let $f(a)$, $a$ in $A$, be a net which satisfies the Cauchy condition for nets; that is, for each positive $\epsilon$ there is an $a_\epsilon$ in $A$ such that $|f(a)-f(a')| < \epsilon$ whenever both $a$ and $a'$ are $\succ a_\epsilon$. We choose successively $a_1, a_2, \cdots$ in $A$ such that $a_n \succ a_{n-1}$ and $|f(a)-f(a')| < 1/n$ whenever $a$ and $a'$ are $\succ a_n$. Then the numbers $f(a_n)$, $n=1, 2, \cdots$ form a Cauchy sequence, so they have a limit $k$. Now to show that the net has $k$ as limit, let $\epsilon$ be a positive number, and pick $n > 2/\epsilon$, so that $|f(a_{n+1})-k| < \epsilon/2$. If $a \succ a_n$, by definition of $a_n$ we have $|f(a_{n+1})-f(a)| < 1/n < \epsilon/2$, so $|f(a)-k| < \epsilon$, completing the proof. This particular theorem is quite convenient, since it provides at once all the assorted forms of the Cauchy criterion for all the various limit processes occurring in advanced calculus and theory of functions of a real variable.

There is, however, one portion of the general theory on which we ask deferment for a few moments. This is the idea of subsequence and the theorems connected with the idea. Before we take this up, we wish to stop to look at some special cases. For we have now generalized the theory of limits from the simple cases of ordinary sequences to something more general, and it is natural that we should pause to see how much new territory we have taken in.

**4. Examples.** In showing that some special kind of limit process is covered by the Moore-Smith theory there is a simple pattern that we often use. The symbol for the limit usually has some notation under the letters lim that indicates, pictorially speaking, in which direction the independent variable is "going." For instance, the limit of a sequence $\lim_{n\to\infty} f_n$ is a limit "as $n$ goes to infinity." To change over to the notation of nets we try to put a partial ordering on the independent variable in such a way that $a \succ b$ shall means that $a$ has "gone further" than $b$. Thus in the case of sequences, $m \succ n$ should mean that $m$ has "gone further toward infinity than $n$ has," that is $m > n$. In the case of a limit $\lim_{x\to c} f(x)$, $x \succ x'$ should mean that $x$ has approached closer to $c$ than $x'$ has, that is $|x-c| < |x'-c|$. In this case, though, we add as usual the requirement that $x$ and $x'$ should not equal $c$.

As a first trial, we consider a real valued function $f$ defined on a set $A$ of real numbers having no finite upper bound, and look at $\lim_{x\to\infty} f(x)$, wherein we of course understand that $x$ is restricted to $A$. According to the pattern described in the preceding paragraph, we define $x \succ x'$ to mean $x > x'$. It is trivially easy to show that $A$ is directed by $\succ$, and that the Moore-Smith limit $\lim_{x \text{ in } A, \succ} f(x)$ is the same as $\lim_{x\to\infty} f(x)$. For a slightly less trivial example we take a double sequence, $f_{m,n}$ $(m, n = 1, 2, \cdots)$ of reals. Since we want the limit as both $m$ and $n$ tend to infinity, we define $(m, n) \succ (m', n')$ to mean that $m > m'$ and $n > n'$. We easily prove that the set of all pairs of positive integers is directed by this relation and that the Moore-Smith limit in this case reduces to $\lim_{m,n\to\infty} f_{m,n}$. The same device applies to sequences with more than two subscripts.

Suppose that $A$ is a set of real numbers, and $p$ a real number such that arbitrarily near it are numbers $a \neq p$ in $A$. Let $f$ be a real valued function on $A$. We wish to show that $\lim_{x\to p} f(x)$ is a special case of the Moore-Smith limit. So we define $x \succ x'$ to mean that neither $x$ nor $x'$ is $p$ and that $|x-p| < |x'-p|$. The Moore-Smith definition reduces to the usual $\epsilon-\delta$-definition except that in it the $\delta$'s are restricted to the form $|x-p|$, $x$ in $A$. This restriction is easily seen to be without effect, so $\lim_{x\to p} f(x)$ is another example of the Moore-Smith theory. The same device can equally well be applied in spaces of any number of dimensions, and in fact in any metric space.

We shall now discuss unordered sums, historically the first step toward the Moore-Smith theory.* Let $X$ be any set, and $f$ a real-valued function on $X$. Even when $X$ has an order-relation, for instance when $X$ is the set of positive integers, we wish to form partial sums with more and more elements, chosen without regard to order. So we let $A$ consist of all finite subsets of $X$, and for each $a_1$ and $a_2$ in $A$ we define $a_1 \succ a_2$ to mean that $a_1$ contains $a_2$, that is all points belonging to $a_2$ belong to $a_1$. This set is clearly directed, since for any $a$ and $b$ in $A$ the set $c$ consisting of all points in $a$ and all points in $b$ satisfies $c \succ a$, $c \succ b$. If $a$ is in $A$ and consists say of $x_1, \cdots, x_k$, we define $S(a)$ to mean $f(x_1) + \cdots + f(x_k)$. This is a net; and $\lim S = k$ means that to each positive

* E. H. Moore, *Definition of limit in general analysis*, Proc. Nat. Acad. Sci., vol. 1, 1915, pp 628–632.

$\epsilon$ corresponds a finite set $a_\epsilon$ such that whenever $a$ is a finite subset of $A$ containing $a_\epsilon$, consisting say of $x_1, \cdots, x_h$, then $|f(x_1)+\cdots+f(x_h)-k|<\epsilon$. It is not difficult to see that when $X$ consists of the positive integers, lim $S$ exists if and only if the series $f(1)+f(2)+\cdots$ is unconditionally convergent, in which case lim $S=\sum_{n=1}^{\infty} f(n)$.

Next we consider a still less traditional looking kind of net. Suppose that $C$ is a plane curve defined by equations $x=x(t)$; $y=y(t)$, $a\leqq t\leqq b$. We wish to give precision to the idea that the length of $C$ is the limit, in some sense, of the lengths of polygons inscribed in $C$ as they acquire more and more vertices. So for $A$ we choose the set of all polygons inscribed in $C$, the word "inscribed" being understood in the following sense. In $[a, b]$ we choose points $t_0=a<t_1<t_2<\cdots<t_{n-1}<t_n=b$, and we join the points $\{x(t_0), y(t_0)\}$, $\{x(t_1), y(t_1)\}$, $\cdots$, $\{x(t_n), y(t_n)\}$ in that order by line segments. The result is a polygon inscribed in $C$. If $P$ and $P'$ are in $A$, we say that $P\succ P'$ if and only if all the vertices of $P'$ are also vertices of $P$. This is easily seen to satisfy (i) of the definition of direction; and if $P'$ and $P''$ are in $A$, we can construct an inscribed polygon whose vertices are all the vertices of $P'$ and all those of $P''$, so that $P\succ P'$ and $P\succ P''$, and part (ii) of the definition is also satisfied. Let $L(P)$ be the length of $P$. This is a net of real numbers. Its limit, if it exists, is called the length of the curve $C$. Since we easily see that $L(P')\geqq L(P'')$ whenever $P'\succ P''$, the general theorem on isotone (or non-decreasing) nets tells us that the limit is the same as the least upper bound, so the length of $C$ is also the least upper bound of the lengths of all polygons inscribed in $C$.

**5. Application to the definition of the integral.** Our next example is more important, because it is one which is often unsatisfactorily treated in text-books. Let $f$ be defined and bounded on an interval $[a, b]$ of real numbers; we wish to define its (Riemann) integral. First we choose numbers $a=x_0<x_1<\cdots<x_n=n$, and other numbers $\xi_1, \cdots, \xi_n$ such that $\xi_i$ is between $x_{i-1}$ and $x_i$. Then we form the sum $\sum_{i=1}^{n}(x_i-x_{i-1})f(\xi_i)$, and we take some kind of limit of this sum. The question is, though, what kind of limit. Sometimes one hears "the limit as the number of subdivisions tends to infinity," with some qualifying remark about the lengths of the subdivisions. But this sum is not a single-valued function of the number of subdivisions. Some kind of extension of the concept of limit is called for. One alternative is to look sternly at the student and say "That's perfectly clear, isn't it?" This is what Mark Kac calls "proof by intimidation." A better way is to widen the theory to cover multiple-valued functions, treating the value of the sum as a multiple-valued function of the length of the longest subinterval $(x_{i-1}, x_i)$. A third way, which we now discuss, is to use the Moore-Smith limit. Let us use the name "partition" for a system of division points $a=x_0<x_1<\cdots<x_n=b$ together with the intermediate points $\xi_1, \cdots, \xi_n$ satisfying $x_{i-1}\leqq\xi_i\leqq x_i$; the "norm" of this partition shall be the greatest of the numbers $x_1-x_0$, $x_2-x_1$, $\cdots$, $x_n-x_{n-1}$. If $P$ is the partition just described, we use it to determine a sum $S(P)=\sum_i f(\xi_i)(x_i-x_{i-1})$. This is a single-valued function

of the partition $P$; in fact, it was just for the purpose of making the sum single-valued that we defined $P$ to consist of both division and intermediate points. Now we wish to express the integral as the limit of $S(P)$ as something happens. This something may be thought of in either of two ways. We may say that we want the longest of the subintervals in the partition to approach zero, or we may say that we want the limit as we cut up the interval $(a, b)$ finer and finer. If we choose the first of these points of view, we would say that for two partitions $P$ and $P'$, the statement $P \succ P'$ should mean that the norm of $P$ is less than the norm of $P'$. If we choose the other point of view, we would say that the statement $P \succ P'$ means that all the points of division $x_0', x_1', \cdots, x_m'$ occurring in $P'$ are among the points $x_0, x_1, \cdots, x_n$ occurring in $P$. In either case, the intermediate points $\xi_i$ are disregarded in defining the order. For the kind of integral we are here discussing it does not matter which of the two definitions of order we choose; if the finite sum has a limit when one of the two definitions of $\succ$ is used it has the same limit when the other is used, and this limit is by definition the (Riemann) integral of $f$. The choice of limiting process is in this case a matter of taste, and likewise in the case of the Riemann integral in higher dimensional spaces. However, it should be mentioned that when we go deeper into analysis and study the Stieltjes integral it does make a difference which definition of $\succ$ we use. It is not that one is "right" and the other "wrong"; the two definitions lead to different integrals, both of which have been investigated.*

**6. Subnets.** Now we come back to the question of subnets, postponed a few pages ago. The ordinary definition of subsequence has a straightforward generalization, but this generalization does not prove perfectly satisfactory. J. L. Kelley has proposed an alternative which may at first seem drastic, since it is not merely another scheme for generalizing the idea of subsequence with which we are all familiar, but proposes that we replace that old and familiar idea with a new one. Suppose that $f_n$, $n=1, 2, \cdots$ is a sequence. We are used to saying that a subsequence of this is a sequence $f_{n_j}$, $n=1, 2, \cdots$, wherein the subscripts $n_j$ are positive integers such that $n_1 < n_2 < n_3 < \cdots$. But if we look at the proofs of the theorems on subsequences, we find that this last condition is not used in its full strength. What is used is the consequence that as $j$ tends to $\infty$, so does $n_j$. Accordingly, Kelley defined a sequence $f_{n_j}$ to be a subsequence of $f_n$ if the $n_j$ are positive integers such that as $j$ tends to $\infty$ so does $n_j$. The generalization to nets is obvious. If $f(a)$, $a$ in $A$ is a net, another net $g(b)$, $b$ in $B$ is a subnet of the first if there is a function $a=a(b)$, $b$ in $B$ such that for all $b$ in $B$, $g(b)=f\{a(b)\}$, and such also that for each $a'$ in $A$ there is a $b'$ in $B$ such that whenever $b \succ b'$ it is also true that $a(b) \succ a'$.

The principal uses of subsequences come by way of the theorem that if a sequence converges, every subsequence converges to the same limit. This gen-

* See, for example, L. M. Graves, *Theory of functions of real variables*, McGraw-Hill, 1946, pp. 260–261.

eralizes at once to nets. As an application, which in spite of its simplicity may serve to hint that our nets may have caught some strange fish in addition to what we wanted and expected, we consider a function $f(x)$, $x$ in $A$ defined on a set $A$ of real numbers, and we assume that $\lim_{x\to c} f(x)$ exists and is equal to $k$. Let $x_1, x_2, x_3, \cdots$ be a sequence of points of $A$ all different from $c$ and having $c$ as limit when $n\to\infty$. We *could* follow the familiar proof that then $f(x_n)$ has $k$ as limit. But we need not; for $f(x_n)$, $n=1, 2, \cdots$ is by our definition a subnet of the net $f(x)$, $x$ in $A$ (wherein as before we define $a\succ a'$ to mean $0<|a-c|<|a'-c|$), so the subnet must have $k$ as limit.

**7. Relations to topology.** So far our discussion has been restricted to topics that an undergraduate might encounter. Now we shall touch briefly on some less elementary uses of Moore-Smith convergence. But first we introduce an abbreviation. If a statement involving members $a$ of a directed set $A$ is true for all $a$ which follow some $a'$ in $A$, we say that the statement is "eventually" true. Thus we have a clear concept to replace the idea of "becomes and remains" often mentioned in elementary texts. It is in fact convenient to use this idea of "eventually" even in the beginning of the study of Moore-Smith limits. If a statement $P$ is eventually true (say for $a\succ a'$) and another statement $Q$ is eventually true (say for $a\succ a''$), we choose $a^*$ such that $a^*\succ a'$ and $a^*\succ a''$; then $P$ and $Q$ are both true for $a\succ a^*$, so the joint statement "$P$ and $Q$" is eventually true.

A topological space is a set $X$ together with a specified collection of subsets of $X$, called the "open sets," such that the union of arbitrarily many open sets is open, the intersection of finitely many open sets is open, and the empty set and $X$ itself are open. A neighborhood of a point $x$ of $X$ is an open set containing $x$. A net $x(a)$, $a$ in $A$ of points of $X$ converges to a point $x'$ of $X$ if for every neighborhood of $x'$, $x(a)$ is eventually in that neighborhood. A point $x'$ is a *cluster point* of the net $x(a)$, $a$ in $A$ if, crudely stated, $x(a)$ "keeps coming back" to every neighborhood of $x'$; precisely,† for each neighborhood $U$ of $x'$ and each $a'$ in $A$, $x(a)$ is in $U$ for some $a\succ a'$.

It is well known that if $X$ is a subset of a euclidean space, either it has all three of the following properties or else it has none of them:

(I) *From every covering of $X$ by open sets it is possible to extract finitely many sets which cover $X$.*

(II) *Every sequence of points of $X$ has a cluster point in $X$.*

(III) *Every sequence of points of $X$ contains a subsequence which converges to a point of $X$.*

These three properties continue to be equivalent for some spaces more general than subsets of euclidean spaces, for example they are equivalent if $X$ is per-

† For this relationship between $U$ and the net $x(a)$, $a$ in $A$, Halmos proposes the terminology "$x$ is frequently in $A\,U$."

fectly separable; but they are not equivalent for all topological spaces. However, as soon as we replace sequences by nets the equivalence is restored. That is, if we write

(II′) *Every net of points of $X$ has a cluster point in $X$,*

(III′) *For every net $x(a)$, $a$ in $A$, of points of $X$ there is a subnet converging to a point of $X$,*

then every topological space $X$ either has all three properties (I), (II′), (III′), or it lacks all three. Suppose that $X$ has property (I), and let $x(a)$, $a$ in $A$ be a net of points of $X$. If this had no cluster point, for each $x$ in $X$ we could find a neighborhood $U(x)$ such that $x(a)$ is eventually out of $U(x)$. Finitely many of these cover $X$, say $U(x_1), \cdots, U(x_n)$. For $a \succ$ a certain $a_{n_j}$, $x(a)$ is not in $U(x_j)$. Choose $a \succ a_1, \cdots, a_n$; then $x(a)$ is out of all the $U(x_j)$ and yet is in $X$, which is impossible. So $X$ has property (II′). Conversely, suppose $X$ lacks property (I). Then there is a collection $K$ of open sets covering $X$ but such that no finite subcollection of $K$ covers $X$. Let $A$ consist of all finite subsets of $K$, and order $A$ by defining $a \succ a'$ to mean that all the sets which belong to $a'$ also belong to $a$. $A$ is directed by $\succ$, since if $a'$ and $a''$ are in $A$, the set $a$ consisting of all sets in $a'$ together with all sets in $a''$ satisfies $a \succ a'$ and $a \succ a''$. For each such $a$, there is a point of $X$ not in any of the sets which constitute $a$, since these sets form a finite subcollection of $K$ and by hypothesis do not cover $X$. Pick such a point and call it $x(a)$. Because $A$ is directed, these $x(a)$, $a$ in $A$ form a net. Now for any $x'$ in $X$ and any set $U$ of the family $K$ which contains $x'$, we let $a'$ be the subset of $K$ consisting of $U$ alone. If $a \succ a'$, then $x(a)$ is outside all the sets in $a$, in particular is outside $U$, so $x'$ is not a cluster point of the net. This holds for all $x'$ in $X$, so $X$ lacks property (II′).

Property (III′) plainly implies (II′). To show the converse,* let $x(a)$, $a$ in $A$ be a net of points of $X$ having $x'$ as cluster point. Let $B$ consist of all the pairs $(U, a)$ in which $U$ is a neighborhood of $x'$ and $a$ is a point of $A$ such that $x(a)$ is in $U$. We order these by defining $(U, a) \gg (U', a')$ to mean that $U$ is contained in $U'$ and $a \succ a'$. This is obviously transitive. If $(U', a')$ and $(U'', a'')$ are in $B$, let $U$ be the intersection of $U'$ and $U''$. This is a neighborhood of $x'$, so there is an $a$ in $A$ such that $a \succ a'$ and $a \succ a''$ and $x(a)$ is in $U$. So by definition $(U, a) \gg (U', a')$ and $(U, a) \gg (U'', a'')$, and $B$ is directed by $\gg$. For each $b = (U, a)$ in $B$ we define $a(b)$ to be the second component of $b$, that is $a(b) = a$. We must prove, first, that $x\{a(b)\}$, $b$ in $B$ is a subnet of $x(a)$, $a$ in $A$, and second, that it converges to $x'$. We do both of these at once, as follows. Let $a'$ be an arbitrary member of $A$ and $U'$ an arbitrary neighborhood of $x'$. There is an $a''$ in $A$ such that $a'' \succ a'$ and $x(a'')$ is in $U'$. Define $b' = (U', a'')$. For every $b = (U, a)$ such that $b \gg b'$, we have first $a \succ a'' \succ a'$, that is $a(b) \succ a'$, whence $x\{a(b)\}$, $b$ in $B$ is a subnet of $x(a)$, $a$ in $A$. And second we have $U$ contained in

* J. L. Kelley, *loc. cit.*

$U'$, so that $x\{a(b)\}$, being in $U$, is also in $U'$, whenever $b \gg b'$; and therefore $x'$ is the limit of $x\{a(b)\}$, $b$ in $B$. This completes the proof of the theorem. It will be noticed that Kelley's definition of subnet is exactly what is needed; under previous definitions of subnet the theorem cannot be established.

When we are dealing with subspaces of finite dimensional spaces, it is quite convenient to be able to maneuver back and forth between property (I), in "Heine-Borel" arguments, and properties (II) and (III), in "Bolzano-Weierstrass" arguments. The proof just completed shows that the same kind of convenience is available even in the most general topological spaces, provided that we agree to use nets instead of sequences and to adopt Kelley's definition of subsequence.

**8. Convergence in partially ordered spaces.** In the earlier pages of this paper we were concerned with sets of real numbers and nets of real numbers. The real numbers have a natural topology, and if we think of the reals as a space of points equipped with this topology the natural generalization is to topological spaces. On the other hand, the reals also constitute a partially ordered, in fact a directed, system. It is thus also reasonable to wonder how much we can deduce from the order relation, and to investigate partially ordered sets and nets of points of partially ordered sets. Suppose then that $X$ is a set of points partially ordered by a relation $\succ$. To save trouble we shall assume that $\succ$ is a *proper* partial ordering, which by definition means that if $x$ and $x'$ are two different points of $X$, the relation $x \succ x'$ and $x' \succ x$ cannot both be true. For example, when in discussing integrals we partially ordered partitions by defining $P \succ P'$ to mean that norm $P \leqq$ norm $P'$, we thereby introduced an improper partial ordering. But if we had instead defined $P \succ P'$ to mean norm $P <$ norm $P'$ the partial ordering would have been proper.

The definitions of upper and lower bounds and of least upper bounds (or suprema) and greatest lower bounds (or infima) can be taken over at once from the real numbers. Among the reals, there are two equivalent ways of defining completeness. The reals are complete in the Cauchy sense—every sequence of reals which satisfies the Cauchy condition is convergent to a real limit. They are also complete in the Dedekind sense—every non-empty set of real numbers which has an upper bound has a least upper bound, and likewise for lower bounds. The former of these is of course much used in metric spaces; it is the other which we now consider. We could of course take it over verbatim, and say that a properly partially ordered set is Dedekind-complete if every non-empty subset which has an upper bound also has a supremum. But a directed set with this property would then have a supremum and an infimum for each two points, and by definition such a set is a lattice. Now lattices are interesting objects to study, as anyone knows who has looked into Birkhoff's book about them.* But at this moment we do not wish to confine our attention to them. So we

* Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. XXV, 1948.

adopt another formulation which makes no difference when we are discussing the reals, but makes just the difference we want in other cases. We say that $X$ is Dedekind-complete* if every non-empty subset $S$ of $X$ which *is directed by* $\succ$ and has an upper bound also has a supremum, and every non-empty subset $S$ of $X$ which is directed by $\succ$ and has a lower bound also has an infimum. Thus for example, the set of circular regions of the plane $(x-x_0)^2+(y-y_0)^2 \leqq r^2$, $r \geqq 0$ is Dedekind-complete if $\succ$ means $\supset$, "contains"; but these circles do not form a lattice.

If $f(a)$, $a$ in $A$ is a net of real numbers, the statement that the limit of $f(a)$ is $k$ can be thus phrased in terms of order: For every real number $m<k$, it is eventually true that $f(a)>m$; and for every real number $n>k$, it is eventually true that $f(a)<n$. This suggests the following definition of convergence in partially ordered sets. If $f(a)$, $a$ in $A$ is a net of points of a partially ordered set $X$, it is convergent if there exist sets $M$ and $N$ in $X$, directed by $\succ$ and $\prec$ respectively, such that the supremum of $M$ is the same as the infimum of $N$, and for every $m$ in $M$ and every $n$ in $N$ it is eventually true that $n \succ f(a) \succ m$. It is easy to show that the limit, if it exists, is unique. From the definition of limit we can proceed to the definition of continuity and to the study of continuous functions. Just where this leads us cannot yet be stated. I have done some studying of partially ordered sets and continuous functions on them, and expect to publish the results soon. But much remains to be done.

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# ON THE RECTANGULAR HYPERBOLA

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**1. Introduction.** By proper choice of axes and unit of scale any rectangular hyperbola may be expressed by the equation $xy=1$, or, parametrically, by $x=t$, $y=t^{-1}$. Choose any three points $A_1$, $A_2$, $A_3$ on the hyperbola such that no side of the triangle is parallel to an asymptote, and designate the parameters of these points by $t_1$, $t_2$, $t_3$, respectively.

If the altitude of the triangle drawn from $A_1$ to the side $A_2A_3$, produced if necessary, makes acute angles $\alpha$ and $\beta$ with the asymptotes of the hyperbola, then the line drawn through $A_1$ which makes acute angles $\beta$ and $\alpha$, respectively, with the asymptotes is called the antiparallel to the altitude with respect to the asymptotes. For brevity in this article, such a line will be termed a slant-line. Thus the antiparallel to the altitude through $A_i$ with respect to the asymptotes of the hyperbola will be called a slant-line and designated by the symbol $H_i$, $(i=1, 2, 3)$.

* [Added in proof.] Since writing this, I have learned that this concept, with isotone sequences in place of directed sets, has been thought of earlier; see M. H. Stone, *A general theory of spectra*, I, Proc. Nat. Acad. Sci. 26 (1940), p. 281, and S. Bochner, *Completely monotone functions*, Duke Math. Jour. 9 (1942), p. 520.