# Diophantine Olympics and World Champions: Polynomials and Primes Down Under

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**1. LET THE GAMES BEGIN: THE OPENING CEREMONIES.** For those who think globally, "down under" may provoke thoughts of Australia—the home of the 2000 Olympic Games. For those who think rationally, "down under" may provoke thoughts of denominators of fractions. In this paper, we hope to provoke both.

In basic diophantine approximation, the name of the game is to tackle the following: How close do integer multiples of an irrational number get to integral values? Specifically, if $\alpha$ is an irrational number and the function $\|\cdot\|$ on $\mathbb{R}$ gives the *distance to the nearest integer* (that is, $\|x\| = \min\{|x-m| : m \in \mathbb{Z}\}$), then the game really is a competition among all integers $n$ to minimize the value $\|\alpha n\|$.

Suppose someone serves us an irrational number $\alpha$. We write $q_1, q_2, q_3, \ldots$ for the (winning) sequence of integers that is able to make it over the following three hurdles:

(i) $0 < q_1 < q_2 < q_3 < \cdots < q_i < \cdots;$
(ii) $\|\alpha q_1\| > \|\alpha q_2\| > \|\alpha q_3\| > \cdots > \|\alpha q_i\| > \cdots;$
(iii) if $q$ is any integer such that $1 \le q < q_n$, $q \ne q_{n-1}$, then $\|\alpha q\| > \|\alpha q_{n-1}\|$.

We now award such a sequence of integers the title: *The world champion approximation sequence for* $\alpha$ or simply say that the $q_n$'s form *the team of world champions for* $\alpha$. Let us momentarily suppress the natural desire to ask the obvious two questions:

- Do world champions exist for each irrational $\alpha$?
- If a world champion sequence does exist, how would we find it?

Instead, let's uncover the connections between the world champions and the Olympic Games "down under".

If we let $p_n$ denote the nearest integer to $\alpha q_n$, then

$$\|\alpha q_n\| = |\alpha q_n - p_n| = q_n\left|\alpha - \frac{p_n}{q_n}\right|,$$

and thus we see the $q_n$ "down under" in the fraction $p_n/q_n$. Suppose that $1 \le q < q_n$ and $\|\alpha q\| = |\alpha q - p|$. Then properties (ii) and (iii) ensure that

$$\left|\alpha - \frac{p}{q}\right| > \left|\alpha - \frac{p_n}{q_n}\right|.$$

Hence we see that $p_n/q_n$ is the *best* rational approximation to $\alpha$ having a $q$ "down under" not exceeding our champion $q_n$.

It is clear that once we know $q_n$, the value of $p_n$ is completely determined: it must be the nearest integer to $\alpha q_n$. Thus in order to find world champion (best) *rational* approximations to $\alpha$, we need focus our attention only on finding the $q_n$'s—that is, the world champions for $\alpha$.

Now what do these world champion sequences look like? Table 1 lists some popular numbers and the first few (in fact *a perfect* 10) terms in their associated world champion sequences ($\mu$ denotes *Mahler's number*: $\mu = 0.1234567891011121314151617181920\ldots$).

TABLE 1

| | $\frac{1+\sqrt{5}}{2}$ | $e$ | $\sqrt[3]{2}$ | $\pi$ | $\mu$ |
|---|---|---|---|---|---|
| $q_1$ | 1 | 1 | 3 | 7 | 73 |
| $q_2$ | 2 | 3 | 4 | 106 | 81 |
| $q_3$ | 3 | 4 | 23 | 113 | 12075796 |
| $q_4$ | 5 | 7 | 27 | 33102 | 12075877 |
| $q_5$ | 8 | 32 | 50 | 33215 | 24151673 |
| $q_6$ | 13 | 39 | 227 | 66317 | 36227550 |
| $q_7$ | 21 | 71 | 277 | 99532 | 169061873 |
| $q_8$ | 34 | 465 | 504 | 265381 | 205289423 |
| $q_9$ | 55 | 536 | 4309 | 364913 | 374351296 |
| $q_{10}$ | 89 | 1001 | 4813 | 1360120 | 579640719 |

Even the casual spectator might not be able to refrain from making several interesting observations. One such observation is that the world champions for $(1+\sqrt{5})/2$ appear to be the complete list of Fibonacci numbers—a perennial favorite sequence among number theory fans. A slightly less visible observation is that the sequences of champions seem to satisfy a recurrence relation of the form: $q_n = a_n q_{n-1} + q_{n-2}$ for some positive integer $a_n$. In fact, as we'll mention again in the next section, this observation holds for all $\alpha$'s.

Our observations show that any sequence of world champions must grow very fast (on the order of exponential growth), and the slowest growing sequence of champions is the Fibonacci sequence where the coefficients $a_n$ are all equal to 1.

All these thoughts inspire the question we pose and consider here. Suppose we are given a sequence of increasing integers. Must they be the complete team of world champions in the eyes of some irrational number? That is, given a sequence, does there always exist a number $\alpha$ that has the given sequence as its sequence of world champions? We know from the previous paragraph that the answer is "no" since slow growing sequences can never make the cut. But what if the sequence were to really work at it, get in shape, and trim down? That is, is it possible that there is always a *subsequence* of any given sequence that contains all the world champions for an irrational number? In particular, do there exist $\alpha$'s for which *all* their world champions are perfect squares? How about perfect cubes? How about primes? After some warm-up's in the next section, we take on these questions and perform some basic routines in the hopes of discovering the thrill of victory. *Let the games begin.*

## 2. WARMING UP: SOME DIOPHANTINE MENTALROBICS.

Our training program begins with a classic feat by Dirichlet from 1842 that still holds the record for being best possible.

**Theorem 1.** *Let $\alpha$ be a real number and let $Q \geq 1$ be an integer. Then there exists an integer $q$ such that $1 \leq q \leq Q$ and*

$$\|\alpha q\| \leq \frac{1}{Q+1}.$$

The pigeonhole principle allows one to give a beautifully executed one line proof of Dirichlet's theorem. In fact, here is the line:

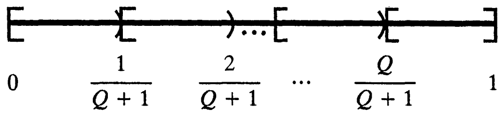


**Figure 1**

which has been partitioned into $Q+1$ subintervals each of length $1/(Q+1)$. We now toss in $Q+2$ points: $\alpha_n = \alpha n - [\alpha n]$ for $n = 0, 1, 2, \dots, Q$ and $\alpha_{Q+1} = \alpha 0 - [-1]$, where $[x]$ denotes the *integer part of* $x$. Thus there must exist two points that land in the same subinterval, and hence there are indices $m_1 < m_2$ such that

$$|\alpha_{m_2} - \alpha_{m_1}| = |\alpha q - p| = \|\alpha q\| \le \frac{1}{Q+1},$$

where $1 \le q \le Q$, and thus we happily find ourselves at the finish line of the proof. ■

Of course if $\alpha$ is an irrational number, then, try as it might, $\|\alpha q\|$ can never equal 0. Therefore by letting $Q$ sprint off to infinity, we immediately have the following.

**Corollary 2.** *If $\alpha$ is an irrational real number, then there exist infinitely many distinct integers $q$ satisfying*

$$\|\alpha q\| < \frac{1}{q}. \tag{2.1}$$

We now want to generate an infinite roster of $q$'s that are fit enough to satisfy the (2.1) challenge. We first write $\alpha = a_0 + \alpha_0$, where $a_0 = [\alpha]$ and $\alpha_0$ denotes the *fractional part of* $\alpha$. Thus $1/\alpha_0 > 1$ and so after a double flip we see

$$\alpha = a_0 + \frac{1}{1/\alpha_0} = a_0 + \frac{1}{a_1 + \alpha_1},$$

where $a_1 = [1/\alpha_0]$ and $\alpha_1$ is the fractional part of $1/\alpha_0$. Since $\alpha$ is irrational, we can repeat this game forever and discover that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where all the $a_n$'s are integers and $a_n > 0$ for all $n > 0$. Such an expansion is the *continued fraction expansion for* $\alpha$, which we write simply as $\alpha = [a_0, a_1, a_2, \dots]$ so as to allow players to print their expansions on the back of their team jerseys. We often hear fans yell out "*partial quotients*!" whenever the $a_n$'s make their appearance. If we decide to call a time-out during the continued fraction game, then our halted process would produce the rational number $[a_0, a_1, \dots, a_n]$, which we denote by $p_n/q_n$ (where $p_n$ and $q_n$ are relatively prime). Those who are true number theory mathletes refer to $p_n/q_n$ as the $n$th *convergent of* $\alpha$. Lagrange, in 1770, thrilled the fans when he showed that for $n > 0$, the $q_n$'s appearing "down under" in the convergents meet all the requirements to be named the world champion sequence for $\alpha$. In fact, the denominators $q_n$ form the *entire* team of world champions for $\alpha$; for the play-by-play details, see [**3**] or [**5**].

By letting some $2 \times 2$ matrices enter into the arena, we can get a better feel for how the $q_n$'s interact with each other. In particular, using induction and some simple linear algebra gymnastics, we can verify the fact that for all $n \geq 0$,

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

If we lob determinants back and forth, we see that $p_n q_{n-1} - p_{n-1} q_n = \pm 1$ and hence we conclude that $q_{n-1}$ and $q_n$ must always be relatively prime. Although consecutive players do not like to share common factors, the matrix product does reveal that any three consecutive members of the team *can* play together in the sense that

$$q_n = a_n q_{n-1} + q_{n-2}. \tag{2.2}$$

These two observations are crucial as sequences prepare for the Olympic Games.

**3. TRAINING TO BE THE BEST: HOW TO SHED UNWANTED TERMS.** Let's now tackle our question: Within every increasing sequence of integers, is there a subsequence that has what it takes to be the complete team of world champions for some irrational number? Sadly, the answer is "no": There is no subsequence of $2, 4, 6, 8, 10, 12, \ldots$ that can be a world champion sequence since, as we've seen at the end of the previous section, consecutive members from a team of world champions must be relatively prime. Thus there can never be a world champion sequence in any sequence of integers for which, from some point onward, all the terms share a common factor. Plainly we should not bother to consider such pathetic sequences that cannot make even the first cut. Unfortunately, there also are 'non-trivial' sequences that never can make it to the Olympic Games. Consider, for example, the sorry sequence beginning with

$$2, 3, 4, 13, 168, 177, 1584, 6396, 83317, 1000128, \ldots. \tag{3.1}$$

The terms in this sequence were carefully recruited so as to satisfy an ever growing list of anonymous congruences (sponsors who wish not to be mentioned here). Notice that in the first ten terms, there does not exist a triple $r < s < t$ such that $t = as + r$ for any positive integer $a$. Thus in view of the line-up from (2.2), we see that the first ten terms in (3.1) do not contain even three players from a world champion team. By our secret recruiting process, this pattern continues for the entire roster of terms in the list—ah, the agony of defeat.

We now bring on some sequences that are better known by the fans. First, let's have sequences generated by polynomial functions take the field.

**Theorem 3.** *Let $f(x)$ be a nonconstant polynomial with integer coefficients whose leading coefficient is positive. Then there exists a sequence of integers $n_1, n_2, n_3, \ldots$ such that $f(n_1), f(n_2), f(n_3), \ldots$ is a complete world champion sequence for some irrational number if and only if there exist integers $n_1$ and $n_2$ such that*

$$0 < f(n_1) < f(n_2) \quad \textit{and} \quad f(n_2) \equiv 1 \bmod f(n_1).$$

*Proof:* If for all $i \geq 1$, $f(n_i) = q_i$ represents a team of world champions of some irrational number, then, as $q_0 = 1$, we must have $0 < q_1 < q_2$ and $q_2 = a_2 q_1 + 1$. Thus we see that $0 < f(n_1) < f(n_2)$ and $f(n_2) \equiv 1 \bmod f(n_1)$.

Conversely, suppose that $0 < f(n_1) < f(n_2)$ and $f(n_2) \equiv 1 \bmod f(n_1)$. We set $a_1 = f(n_1)$ and let $a_2$ be the positive integer such that $f(n_2) = a_2 f(n_1) + 1$. Let's now assume that $n_1, n_2, \ldots, n_I$ have been defined, are warmed up, and are ready to play, where $I \geq 2$. From the benches, we now select a positive integer $c_{I+1}$ that is so large that if $n_{I+1}$ is defined by $n_{I+1} = c_{I+1} f(n_I) + n_{I-1}$, then $f(n_I) < f(n_{I+1})$. Such integers $c_{I+1}$ exist since $f(x)$ tends to infinity as $x$ pumps up. Notice that

$$f(n_{I+1}) = f(c_{I+1} f(n_I) + n_{I-1}) \equiv f(n_{I-1}) \bmod f(n_I).$$

Therefore there exists a positive integer, say $a_{I+1}$, such that

$$f(n_{I+1}) = a_{I+1} f(n_I) + f(n_{I-1}).$$

Thus if we let $q_i = f(n_i)$ for all $i \geq 1$, then the $q_i$'s are the team of world champions for the irrational number $[0, a_1, a_2, a_3, \ldots]$ and we've won the game. ■

As an immediate consequence of Theorem 3, we see that there are irrational numbers whose world champion sequences contain only perfect squares; there are other irrationals whose world champion sequences contain only cubes, or for that matter, any power. Consider, for example, the number

$$\begin{aligned} \alpha &= 0.75994513286162937685173771612208505499708539223856599896303745553\ldots \\ &= [0,1,3,6,29,739,538810,290287122557,842666130962812439208 95,\ldots]. \end{aligned}$$

The first few world champions for $\alpha$ are given in Table 2.

TABLE 2

| $n$ | $q_n$ |
|---|---:|
| 0. | 1 |
| 1 | 1 |
| 2 | 4 |
| 3 | 25 |
| 4 | 729 |
| 5 | 538756 |
| 6 | 290287121089 |
| 7 | 842661309628124284 3329 |
| 8 | 710086208271835755974856388051748579644158 0544 |

It is immediately apparent to any calculator that all the entries in the right column are indeed perfect squares—thus those "... "s can be replaced by real, red-blooded digits to produce an $\alpha$ that has only perfect square world champions!

Another potentially amusing example can be found if we take on the polynomial $f(x) = Ax + 1$, for some integer $A > 0$. Selecting $n_1 = 1$ and $n_2 = A + 1$, we immediately conclude that there exist real numbers $\alpha$ such that each element $q_n$ of its world champion sequence satisfies $q_n \equiv 1 \bmod A$. Thus there are complete teams of world champions agile enough to dodge having a particular factor. In this spirit of dodging factors, we now describe a winning strategy for a similar game involving the ever popular and timeless team of prime numbers.

**Theorem 4.** *There exist irrational numbers $\alpha$ that have only prime numbers as their world champions.*

*Proof:* As always, we set $q_0 = 1$. Next, we set $q_1 = 2$ and $q_2 = 3$, so $a_1 = 2$ and $a_2 = 1$. Suppose now that the primes $q_1 < q_2 < \cdots < q_I$ have all been defined for $I \geq 2$. We now consider the arithmetic progression $\{Aq_I + q_{I-1} : A = 1, 2, \ldots\}$, and apply another important result of Dirichlet, which states that if $r$ and $s$ are relatively prime positive integers, then the arithmetic progression $r + s, 2r + s, 3r + s, \dots$ contains infinitely many primes (see [4] for an instant re-play of this major AP upset). Thus we know that there exists a positive integer $A$, call it $a_{I+1}$, such that $a_{I+1}q_I + q_{I-1}$ is a prime, let's name it $q_{I+1}$. Therefore if we let $\alpha = [0, a_1, a_2, a_3, \dots]$, then its world champion sequence consists solely of prime numbers—this places us right in the middle of the winner's circle. ∎

As an illustration, we now introduce the number

$$\alpha = 0.385477827323240651531341006254937728817527208323735535817422071765 82\dots$$
$$= [0, 2, 1, 1, 2, 6, 2, 10, 18, 20, 16, \dots]$$

and its first ten world champions:

TABLE 3

| $n$ | $q_n$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 5 |
| 4 | 13 |
| 5 | 83 |
| 6 | 179 |
| 7 | 1873 |
| 8 | 33893 |
| 9 | 679733 |
| 10 | 10909621 |

One doesn't really require Olympic-like factorization skills to verify that the champions in Table 3 are all prime numbers.

Finally we remark that in the proofs of Theorems 3 and 4 we are able, without the use of steroids, to make the world champions grow as fast as we wish. This observation in turn implies that the partial quotients, $a_n$, in the continued fraction expansion for the associated irrational $\alpha$ can grow at record breaking speeds (in fact we saw this particular event live during our perfect square example). It turns out that if there is a subsequence of $a_n$'s that grow amazingly fast, then the associated number $\alpha$ must, in fact, be a transcendental number. This remarkable fact follows from a record setting 1844 result due to Liouville, who showed that algebraic numbers cannot be approximated too well by rational numbers whose sizes "down under" are modest; see [3] or [5] for flashbacks to this history-making score. Thus we close our games with the commentary that we may find *transcendental* numbers that accomplish all that is demanded of them in both Theorems 3 and 4. In fact, the $\alpha$ debuting just after Theorem 3 is transcendental.

**4. THE 2004 OLYMPICS: CAN OUR FAVORITE CHAMPS ARISE FROM THE BAD?** With any pursuit—athletic or intellectual—we should always look ahead toward challenges for future thrill-seekers. Thus in our closing ceremonies we look ahead to the next Olympic Games and hope to inspire others to go for the gold.

The $\alpha$'s that are able to give birth to interesting world champions as described in Theorems 3 and 4 can have arbitrarily large $a_n$ values in their continued fraction expansions. Is it possible to find an $\alpha$ that satisfies either theorem and for which all its partial quotients are *bounded*? Numbers $\alpha$ having bounded partial quotients are known as *badly approximable numbers*. This title has been awarded since it can be shown that a number $\alpha$ has bounded partial quotients if and only if (2.2) cannot be improved in the sense that there exists a constant $c = c(\alpha) > 0$ such that $\frac{c}{q} < \|\alpha q\|$ for *all* integers $q$; see [3] or [5] for the proof of why these numbers deserve the name *bad*.

A completely new game plan would be required if one wanted to attempt to show that there are badly approximable numbers whose world champions satisfy Theorems 3 or 4. In the proof of Theorem 3, the partial quotients are born and bred to race off at record speeds to infinity. Is there another training technique that prevents the partial quotients from running away?

To adopt the strategy of the proof of Theorem 4, one would, at the very least, need to know that there exists a constant $C$ such that each arithmetic progression of the form $\{Ar + s : A = 1, 2, \dots\}$, where $\gcd(r, s) = 1$, contains a prime $p \le Cr$. Such an assertion appears to be ridiculously optimistic as the world record-holder in this direction is Heath-Brown [2] who in 1992 nearly caused a riot among fans when he produced the *startling* result that every arithmetic progression $\{Ar + s : A = 1, 2, \dots\}$ with $\gcd(r, s) = 1$ contains a prime $p \le Cr^{5.5}$. The previous world champion exponent was a whopping 13.5, found by Chen and Liu [1] back in 1989. The exponent on $r$ is known as *Linnik's constant*.

These closing remarks may invite both the number theory athlete and fan to train and take on the challenge of showing that there do not exist badly approximable numbers whose world champions are all perfect squares or are all primes—two potentially difficult goals, but certainly in the true tradition of the Olympic spirit.

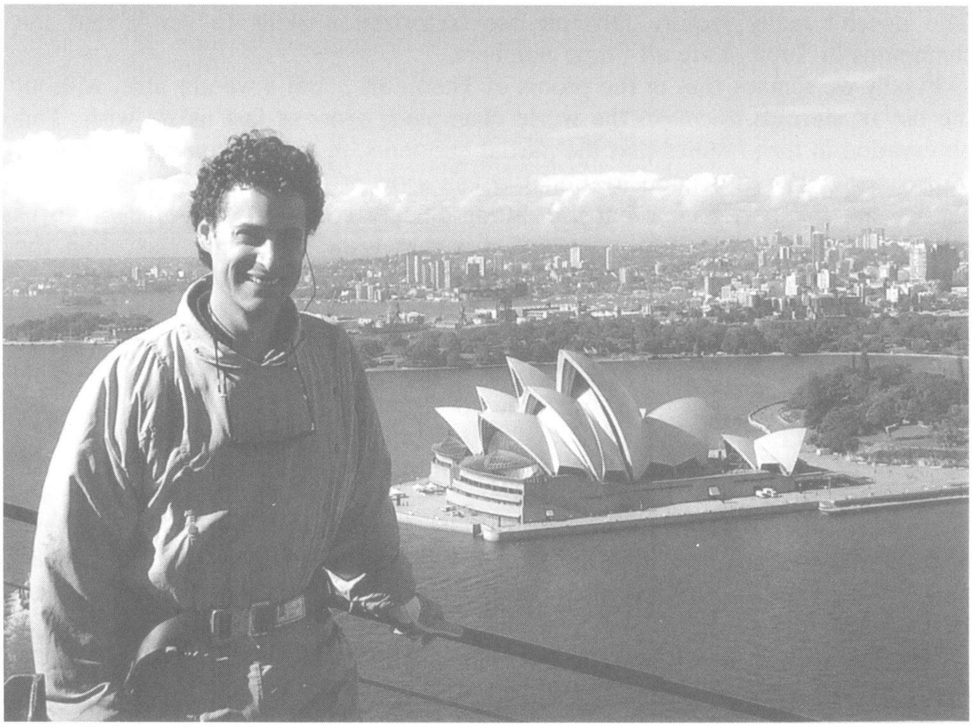

**Figure 2.** The author overlooking Sydney—the site of the 2000 Olympic Games, where he was inspired to consider turning sequences into world champions.

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## FUZZY LOGIC

When we have looked upon each hand
We say we have dichotomied
How complicated then must be
The logic of a centipede.

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