Close Encounters with the Stirling Numbers of the Second Kind

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Let $n = 4$, and consider the terms in row n of Pascal's triangle, with alternating signs: $(1, -4, 6, -4, 1)$. Treat this list as a vector and take its scalar product with a vector of consecutive integer squares:

$$(1, -4, 6, -4, 1) \cdot (0, 1, 4, 9, 16) = 0.$$

Next, try cubes:

$$(1, -4, 6, -4, 1) \cdot (0, 1, 8, 27, 64) = 0.$$

So far, this is getting us nothing. Vectors of first powers and of zeroth powers also give scalar products of zero. We get something more when we try fourth powers:

$$(1, -4, 6, -4, 1) \cdot (0, 1, 16, 81, 256) = 24,$$

which is equal to $n!$.

These are all instances of the strange evaluation,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = \begin{cases} 0 & \text{if } m < n, \\ (-1)^n n! & \text{if } m = n, \end{cases} \tag{1}$$

which we see from time to time in books and articles. For example, Katsuura [**12**] gives the following theorem, extending (1) a little bit: For any two real or complex numbers x, y and for any positive integers m and n,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (xk + y)^m = \begin{cases} 0 & \text{if } m < n, \\ (-1)^n x^n n! & \text{if } m = n. \end{cases} \tag{2}$$

From this form we see that the mth powers do not need to be of consecutive integers; the identity holds for the mth powers of consecutive terms in any arithmetic sequence. This, indeed, is a very strange result!

Is it just a curious fact, or is there something bigger behind it? Also, what happens when $m > n$? The theorem obviously deserves further elaboration. Therefore, we want to fill this gap now and also to provide some related historical information.

Identity (2) is not new. It appears in a more general form in H. W. Gould's *Combinatorial Identities* [**7**]. Namely, if $f(t) = c_0 + c_1 t + \cdots + c_m t^m$ is a polynomial of degree m, then Gould's entry (Z.8) says that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = \begin{cases} 0 & \text{if } m < n, \\ (-1)^n n!\, c_n & \text{if } m = n, \end{cases} \tag{3}$$

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which implies (2). As Gould writes on p. 82: "Relation (Z.8) is very useful; we have numerous interesting cases by choosing $f(t)\ldots$." Identity (2) was later rediscovered by Ruiz [**16**], who proved it by induction.

Here is a simple observation (made also by Katsuura)—expanding the binomial $(xk+y)^m$ in (2) and changing the order of summation, we find that (2) is based on the more simple identity (1) (which, by the way, is entry (1.13) in [**7**]). In his paper [**8**] Gould provides a nice and thorough discussion of identity (1), calling it *Euler's formula,* as it appears in the works of Euler on nth differences of powers. See also Schwatt's book [**17**, pp. 18–19, 48].

The mystery of identity (1) is revealed by its connection to the Stirling numbers. An old result in classical analysis (also discussed in [**8**]) says that

$$(-1)^n n!\, S(m,n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m, \tag{4}$$

where $S(m,n)$ are the *Stirling numbers of the second kind* [**2, 5, 8, 9, 11, 19**]. They have the property $S(m,n)=0$ when $m<n$, and $S(m,m)=1$. We can define these numbers in combinatorial terms: $S(m,n)$ counts the number of ways to partition a set of m elements into n nonempty subsets. Thus we can read $S(m,n)$ as "m subset n." An excellent combinatorial treatment of the Stirling numbers can be found in [**9**]. Pippenger's recent article in this MAGAZINE [**14**] mentions, among other things, their probabilistic interpretation.

A simple combinatorial argument ([**9**, p. 259]) provides the important recurrence

$$S(m,n) = nS(m-1,n) + S(m-1,n-1), \tag{5}$$

valid for $m>0$ and for all integers n, which together with the initial conditions $S(0,0)=1$ and $S(0,n)=0$ for $n\neq 0$, gives an alternative definition for these numbers. Using this recurrence we can compute

$$\begin{aligned} S(n+1,n) &= nS(n,n) + S(n,n-1) \\ &= n + S(n,n-1) \\ &= n + (n-1) + S(n-1,n-2) \\ &= n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}; \end{aligned}$$

i.e., $S(n+1,n) = \frac{n(n+1)}{2}$; and now from (4) we find

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{n+1} = \frac{(-1)^n n}{2}(n+1)!,$$

which extends (1) to the case of $m=n+1$. A proof of (5) using finite differences is presented in [**8**].

The alternative notation

$$S(m,n) = \left\{ {m \atop n} \right\}$$

suggested in 1935 by the Serbian mathematician Jovan Karamata (see [**9**, p. 257]) fits very well with the combinatorial interpretation. With this notation, the recursion (5) becomes

$$\left\{ {m \atop n} \right\} = n \left\{ {m-1 \atop n} \right\} + \left\{ {m-1 \atop n-1} \right\} \quad (n \le m),$$

which parallels the well-known property of binomial coefficients

$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}.$$

With the help of equation (4) we can fill the gap in (2) for $m > n$. Namely, we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (xk + y)^m = (-1)^k n! \sum_{j=n}^{m} \binom{m}{j} x^j y^{m-j} S(j, n).$$

For the proof we just need to expand $(xk + y)^m$, change the order of summation, and apply (4).

Identity (3) has a short and nice extension beyond polynomials. The representation

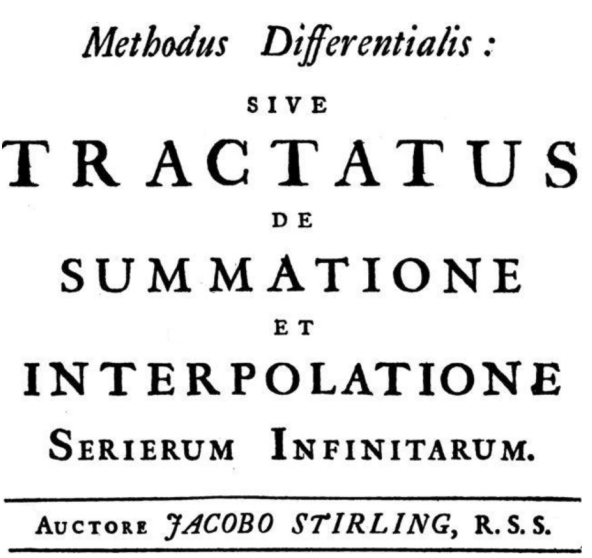
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = (-1)^n n! \sum_{m=0}^{\infty} c_m S(m, n) \tag{6}$$

is true for any $n \geq 0$ and any function $f(t) = c_0 + c_1 t + \cdots$ that is analytic on a disk with radius $R > n$. To prove this, we multiply (4) by c_m and sum for m from zero to infinity.

A combinatorial proof of (4) based on the combinatorial definition of $S(m, n)$ can be found in [**5**, pp. 204–205]; a proof based on finite differences is given in [**11**, p. 169; see also 177–178 and 189–190]. We shall present here two proofs of (1) and (4). For the first one we shall visit the birthplace of the Stirling numbers.

James Stirling and his table

The name "Stirling numbers" comes from the Danish mathematician Niels Nielsen (1865–1931). On p. 68 of his book [**13**] Nielsen attributed these numbers to James Stirling, a Scottish mathematician (1692–1770), who worked on Newton series.


Methodus Differentialis:

SIVE

TRACTATUS

DE

SUMMATIONE

ET

INTERPOLATIONE

SERIERUM INFINITARUM.

AUCTORE *JACOBO STIRLING*, R. S. S.

Figure 1 Part of the front page of Stirling's book

Stirling studied at Oxford, then went to Italy for political reasons and almost became a professor of mathematics in Venice. In 1718 he published through Newton a paper titled *Methodus Differentialis Newtoniana Illustrata.* In 1725 Stirling returned to England and in 1730 published his book *Methodus Differentialis* (The Method of Differences) [**18**]. The book was written in Latin, as were most scientific books of that time. An annotated English translation was published recently by Ian Tweddle [**19**].

At that time mathematicians realized the importance of series expansion of functions, and various techniques were gaining momentum. A *Newton series* is an expansion of a function, say f, in terms of the *difference polynomials,* $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = z(z-1)$, $P_3(z) = z(z-1)(z-2)$, and in general $P_k(z) = z(z-1)\cdots(z-k+1)$. That is,

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k z(z-1)(z-2)\cdots(z-k+1) \\ &= a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \cdots . \end{aligned} \tag{7}$$

The difference polynomials are also called *falling powers,* and they are a basis for the space of polynomials. In this way a Newton series resembles a Taylor series, which is an expansion of f in terms of another basis, the *power polynomials* $p_k(z) = z^k$, $k = 0, 1, \ldots$. The attention paid to both series raised the question of the relationships between the difference polynomials and the power polynomials.

At the beginning of his book Stirling studied carefully the coefficients A_m^n in the representations

$$\begin{aligned} z^m = A_1^m z + A_2^m z(z-1) + A_3^m z(z-1)(z-2) \\ + \cdots + A_m^m z(z-1)\cdots(z-m+1) \end{aligned} \tag{8}$$

where $m = 1, 2, \ldots$. On p. 8 he presented a table containing many of these coefficients, reproduced here as FIGURE 2.

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

Figure 2 Stirling's first table

In the table m changes horizontally, left to right, and n changes vertically, from top to bottom. Therefore, by following the columns of the table we find

$$\begin{aligned}
z &= z,\\
z^2 &= z + z(z-1),\\
z^3 &= z + 3z(z-1) + z(z-1)(z-2),\\
z^4 &= z + 7z(z-1) + 6z(z-1)(z-2) + z(z-1)(z-2)(z-3),\\
z^5 &= z + 15z(z-1) + 25z(z-1)(z-2) + 10z(z-1)(z-2)(z-3)\\
&\quad + z(z-1)(z-2)(z-3)(z-4), \text{ etc.}
\end{aligned}$$

The coefficients A_k^m are exactly the numbers which we call today Stirling numbers of the second kind. For completeness, we add to this sequence also $A_0^0 = 1$ and $A_0^m = 0$ when $m > 0$. The following is true.

THEOREM 1. *Let the coefficients A_n^m be defined by the expansion* (8). *Then*

$$A_n^m = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m, \tag{9}$$

and the right side is zero when $n > m$.

It is not obvious that (8) implies (9). For the proof of the theorem we need some preparation.

Stirling's technique for computing this table is presented on pp. 24–29 in Ian Tweddle's translation [**19**]. As Tweddle comments on p. 171, had Stirling known the recurrence relation (5), the computation of the table would have been much easier.

Newton series and finite differences

The theory of Newton series, like (7), also resembles the theory of Taylor series. First of all, one needs to find a formula for the coefficients a_k. In the case of Taylor series, the function f is expanded on the power polynomials and the coefficients are expressed in terms of the higher derivatives of the function evaluated at zero. In the case of Newton series, instead of derivatives, one needs to use finite differences. This is suggested by the very form of the series, as the function is expanded now on the difference polynomials.

For a given function $f(z)$ we set

$$\Delta f(z) = f(z+1) - f(z).$$

Then

$$\begin{aligned}
\Delta^2 f(z) &= \Delta(\Delta f)(z) = f(z+2) - 2f(z+1) + f(z),\\
\Delta^3 f(z) &= \Delta(\Delta^2 f)(z) = f(z+3) - 3f(z+2) + 3f(z+1) - f(z), \text{ etc.}
\end{aligned}$$

We notice the binomial coefficients appearing here with alternating signs. Following this pattern we arrive at the representation

$$\Delta^n f(z) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(z+k).$$

In particular, with $z = 0$,

$$\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k). \tag{10}$$

We use this formula now to compute the coefficients a_k in the Newton series (7). With $z = 0$ we see that $a_0 = f(0)$. A simple computation shows that $\Delta P_k(z) = k P_{k-1}(z)$, $k \geq 1$ and so

$$\Delta f(z) = a_1 + 2a_2 z + 3a_3 z(z-1) + \cdots,$$

which yields $a_1 = \Delta f(0)$. Also,

$$\Delta^2 f(z) = 2a_2 + 2 \cdot 3a_3 z + 3 \cdot 4z(z-1) + \cdots$$

and $2a_2 = \Delta^2 f(0)$. Continuing this way we find $3!\,a_3 = \Delta^3 f(0)$, $4!\,a_4 = \Delta^4 f(0)$, etc. The general formula is $k!\,a_k = \Delta^k f(0)$, $k = 0, 1, \ldots$. Thus (7) becomes

$$f(z) = \sum_{k=0}^{\infty} \frac{\Delta^k f(0)}{k!} z(z-1)(z-2)\cdots(z-k+1). \tag{11}$$

Proof of Theorem 1. Take $f(z) = z^m$ in (11) to obtain (in view of (10))

$$z^m = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m \right\} z(z-1)(z-2)\cdots(z-n+1). \tag{12}$$

Comparing this to (8) yields the representation (9). Note also that the series (12) truncates, as on the left-hand side we have a polynomial of degree m. The summation on the right-hand side stops with $n = m$, and

$$\frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^m = 1. \qquad \blacksquare$$

This is not the end of the story, however. In the *curriculum vitae* of the Stirling numbers there is another remarkable event.

Grünert's polynomials

Amazingly, the same Stirling numbers appeared again, one hundred years later, in a very different setting. They appeared in the work [**10**] of the German mathematician Johann August Grünert (1797–1872), professor at the University of Greifswald, Germany. He taught there from 1833 until his death. Grünert, a student of Pfaff and Gauss, was interested in many topics, not only in mathematics, but also in physics. He wrote a number of books on such diverse subjects as conic sections, the loxodrome, optics, and the solar eclipse. Some of his books, including *Optische Untersuchungen* (Studies in Optics) and *Theorie der Sonnenfinsternisse* (Theory of the Solar Eclipse) are available now as Google books on the Internet. In 1841 Grünert started to edit and publish the highly respected *Archiv der Mathematik und Physik* (known also as "Grünert's Archiv"). His biography, written by his student Maximus Curtze, appeared in volume 55 of that journal.

Grünert came to the numbers $S(m, n)$ by repeatedly applying the operator $x\frac{d}{dx}$ to the exponential function e^x. This procedure generates a sequence of polynomials

$$\begin{aligned}
x\frac{d}{dx}e^x &= xe^x,\\
\left(x\frac{d}{dx}\right)^2 e^x &= (x^2+x)e^x,\\
\left(x\frac{d}{dx}\right)^3 e^x &= (x^3+3x^2+x)e^x,\\
&\vdots\\
\left(x\frac{d}{dx}\right)^m e^x &= (B_0^m + B_1^m x + B_2^m + \cdots + B_m^m x^m)e^x, \qquad (13)
\end{aligned}$$

with certain coefficients B_k^m. We shall see that these coefficients are exactly the Stirling numbers of the second kind. This fact follows from the theorem below.

THEOREM 2. (GRÜNERT) *Let the coefficients B_n^m be defined by equation* (13). *Then*

$$B_n^m = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m. \qquad (14)$$

Proof. From the expansion

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!},$$

we find

$$\left(x\frac{d}{dx}\right)^m e^x = \sum_{k=0}^{\infty}\frac{k^m x^k}{k!},$$

for $m = 0, 1, \ldots,$ and from (13),

$$\begin{aligned}
B_0^m + B_1^m x + B_2^m x^2 + \cdots + B_m^m x^m &= e^{-x}\sum_{k=0}^{\infty}\frac{k^m x^k}{k!}\\
&= \left\{\sum_{j=0}^{\infty}\frac{(-1)^j x^j}{j!}\right\}\left\{\sum_{k=0}^{\infty}\frac{k^m x^j}{k!}\right\}. \qquad (15)
\end{aligned}$$

Multiplying the two power series on the right-hand side yields

$$B_0^m + B_1^m x + B_2^m x^2 + \cdots + B_m^m x^m = \sum_{n=0}^{\infty}x^n\left\{\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m\right\},$$

and again, comparing coefficients we see that the series on the right-hand side is finite and (14) holds. The theorem is proved! ■

From (14), (9), and (4) we conclude that $A_n^m = B_n^m = S(m, n)$.

Intermediate summary and the exponential polynomials

We summarize the story so far.

The coefficients A_n^m defined by the representation (8) are the same as the coefficients B_n^m defined by equation (13), and also the same as the Stirling numbers of the second kind $S(m, n)$:

$$\begin{aligned} A_n^m &= B_n^m = S(m, n) \\ &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m. \end{aligned}$$

Also, $S(m, n) = 0$ when $m < n$, and $S(n, n) = 1$. In particular, this proves (1).

The Stirling numbers of the second kind are used in combinatorics, often with the notation $S(m, n) = \left\{ {m \atop n} \right\}$. The number $S(m, n)$ gives the number of ways by which a set of m elements can be partitioned into n nonempty subsets. Thus $\left\{ {m \atop n} \right\}$ is naturally defined for $n \le m$ and $\left\{ {n \atop n} \right\} = 1$. When $m < n$, $\left\{ {m \atop n} \right\} = 0$. The numbers $\left\{ {m \atop n} \right\}$ equal A_n^m because they satisfy (4) as proven in [**5**].

The polynomials

$$\phi_n(x) = S(n, 0) + S(n, 1)x + \cdots + S(n, n)x^n,$$

$n = 0, 1, \ldots,$ appearing in Grünert's work, are called *exponential polynomials*. They have been rediscovered and used by several authors. These polynomials are defined by equation (13), i.e.,

$$\phi_n(x) = e^{-x}\left(x\frac{d}{dx}\right)^n e^x \tag{16}$$

or by the generating function (see [**2**]),

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty}\frac{\phi_n(x)}{n!}t^n.$$

Here are the first five of them.

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= x \\ \phi_2(x) &= x^2 + x \\ \phi_3(x) &= x^3 + 3x^2 + x \\ \phi_4(x) &= x^4 + 6x^3 + 7x^2 + x \end{aligned}$$

A short review of these polynomials is given in [**2**]. Replacing x by ax in (16), where a is any constant, we see that (16) can be written as

$$\left(x\frac{d}{dx}\right)^n e^{ax} = \phi_n(ax)e^{ax}. \tag{17}$$

This form is useful in some computations.

The exponential generating function for $S(m, n)$

For any integer $n \geq 0$, let us expand the function $f(x) = (e^x - 1)^n$ in a Taylor series about $x = 0$ (i.e., a Maclaurin series):

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m.$$

For this purpose we first write

$$(e^x - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^{kx},$$

and then, according to (14) or (4) we compute $f^{(m)}(0)$,

$$\left(\frac{d}{dx}\right)^m (e^x - 1)^n \bigg|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m = n!\, S(m, n).$$

Therefore,

$$\frac{1}{n!}(e^x - 1)^n = \sum_{m=0}^{\infty} S(m, n) \frac{x^m}{m!}. \tag{18}$$

This is the exponential generating function for the Stirling numbers of the second kind $S(m, n)$. The summation, in fact, can be limited to $m \geq n$, as $S(m, n) = 0$ when $m < n$. Equation (18) is often used as the definition of $S(m, n)$.

Euler and the derivatives game

Let $|x| < 1$. We want to show that the numbers $S(m, n)$ naturally appear in the derivatives

$$\left(x \frac{d}{dx}\right)^m \frac{1}{1-x} = \sum_{n=0}^{\infty} n^m x^n, \tag{19}$$

where $m = 0, 1, \ldots$. To show this we first write

$$\frac{1}{1-x} = \int_0^{\infty} e^{-(1-x)t}\, dt = \int_0^{\infty} e^{xt} e^{-t}\, dt.$$

Then, in view of (17),

$$\begin{aligned}
\left(x \frac{d}{dx}\right)^m \frac{1}{1-x} &= \int_0^{\infty} \phi_m(xt) e^{xt} e^{-t}\, dt \\
&= \sum_{n=0}^{m} S(m, n) x^n \int_0^{\infty} t^n e^{-(1-x)t}\, dt \\
&= \sum_{n=0}^{m} S(m, n) x^n \frac{n!}{(1-x)^{n+1}} \\
&= \frac{1}{1-x} \sum_{n=0}^{m} S(m, n) n! \left(\frac{x}{1-x}\right)^n. \qquad (20)
\end{aligned}$$

For the third equality we use the well-known formula (which defines the Laplace transform of t^α)

$$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = \int_0^\infty t^\alpha e^{-st}\,dt.$$

Introducing the polynomials

$$\omega_m(z) = \sum_{n=0}^{m} S(m,n)n!\,z^n,$$

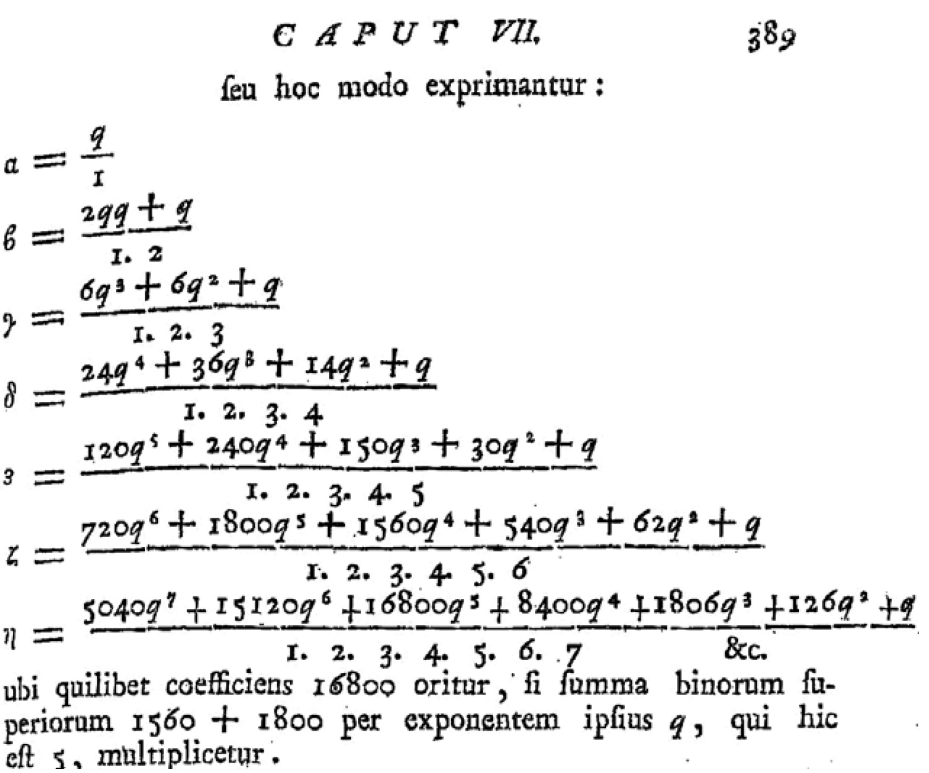
we can write (20) in the form

$$\sum_{n=0}^{\infty} n^m x^n = \frac{1}{1-x}\omega_m\left(\frac{x}{1-x}\right). \tag{21}$$

Thus we have

$$\begin{aligned}
\omega_0(x) &= 1,\\
\omega_1(x) &= x,\\
\omega_2(x) &= 2x^2 + x,\\
\omega_4(x) &= 24x^4 + 36x^3 + 14x^2 + x, \text{ etc.}
\end{aligned}$$

The polynomials ω_n can be seen on p. 389, in Part 2, Chapter VII of Euler's book [**6**].



CAPUT VII. 389

ſeu hoc modo exprimantur:

$$\alpha = \frac{q}{1}$$

$$\beta = \frac{2qq + q}{1.\ 2}$$

$$\gamma = \frac{6q^3 + 6q^2 + q}{1.\ 2.\ 3}$$

$$\delta = \frac{24q^4 + 36q^3 + 14q^2 + q}{1.\ 2.\ 3.\ 4}$$

$$\varepsilon = \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1.\ 2.\ 3.\ 4.\ 5}$$

$$\zeta = \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1.\ 2.\ 3.\ 4.\ 5.\ 6}$$

$$\eta = \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7} \quad \&c.$$

ubi quilibet coefficiens 16800 oritur, ſi ſumma binorum ſuperiorum 1560 + 1800 per exponentem ipſius q, qui hic eſt 5, multiplicetur.

Figure 3 The geometric polynomials in Euler's work

Essentially, Euler obtained these polynomials by computing the derivatives (19) directly. We shall see now how all this can be done in terms of exponentials.

Here is a good exercise. Let us expand the function

$$f(t) = \frac{1}{\mu e^{\lambda t} + 1}$$

in Maclaurin series (λ, μ are two parameters). We need to find the higher derivatives of f at zero. Assuming for the moment that $|\mu e^{\lambda t}| < 1$ we use the expansion

$$\frac{1}{\mu e^{\lambda t} + 1} = \frac{1}{1 - (-\mu e^{\lambda t})} = \sum_{n=0}^{\infty} (-\mu)^n e^{\lambda t n}$$

From this

$$\left(\frac{d}{dt}\right)^m \frac{1}{\mu e^{\lambda t} + 1} = \lambda^m \sum_{n=0}^{\infty} (-\mu)^n n^m e^{\lambda t n}$$

and in view of (21)

$$\frac{1}{\mu e^{\lambda t} + 1} = \frac{1}{\mu e^{\lambda t} + 1} \omega_m \left(\frac{-\mu e^{\lambda t}}{\mu e^{\lambda t} + 1}\right),$$

so that

$$\left(\frac{d}{dt}\right)^m \frac{1}{\mu e^{\lambda t} + 1}\bigg|_{t=0} = \frac{\lambda^m}{\mu + 1} \omega_m \left(\frac{-\mu}{\mu + 1}\right), \tag{22}$$

which yields the desired representation

$$\frac{1}{\mu e^{\lambda t} + 1} = \frac{1}{\mu + 1} \sum_{m=0}^{\infty} \lambda^m \omega_m \left(\frac{-\mu}{\mu + 1}\right) \frac{t^m}{m!}.$$

In particular, with $\lambda = \mu = 1$,

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} \omega_m \left(\frac{-1}{2}\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} S(m, n) n! \, \frac{(-1)^n}{2^n} \right\} \frac{t^m}{m!}. \tag{23}$$

The polynomials ω_m appeared in the works of Euler, but they do not carry his name. In [**8**] and [**17**], they are used to evaluate the series on the right-hand side of (19) in terms of Stirling numbers. These polynomials were studied in [**4**] and called *geometric polynomials,* because of their obvious relation to the geometric series. It was shown in [**4**] that ω_m participate in a certain *series transformation formula.* In [**3**] the geometric polynomials were used to compute the derivative polynomials for $\tan x$ and $\sec x$.

One can write (21) in the form

$$\sum_{n=0}^{\infty} n^m x^n = \frac{A_m(x)}{(1 - x)^{m+1}}$$

where A_m are polynomials of degree m. These polynomials are known today as Eulerian polynomials and their coefficients are the Eulerian numbers [**5, 9**].

At the same time, there is a sequence of interesting and important polynomials carrying the name *Euler polynomials.* These are the polynomials $E_m(x)$, $m = 0, 1, \ldots,$ defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}. \tag{24}$$

Using (23) we write, as in (15)

$$\begin{aligned} \frac{2e^{xt}}{e^t + 1} &= \left\{ \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \right\} \left\{ \sum_{k=0}^{\infty} \omega_k \left(\frac{-1}{2} \right) \frac{t^k}{k!} \right\} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \left\{ \sum_{k=0}^{m} \binom{m}{k} \omega_k \left(\frac{-1}{2} \right) x^{m-k} \right\}. \end{aligned} \tag{25}$$

Comparing (24) and (25) yields

$$E_m(x) = \sum_{k=0}^{m} \binom{m}{k} \omega_k \left(\frac{-1}{2} \right) x^{m-k}$$

with

$$\begin{aligned} E_m(0) &= \omega_m \left(\frac{-1}{2} \right) \\ &= \sum_{n=0}^{m} S(m, n) n! \, \frac{(-1)^n}{2^n}. \end{aligned}$$

Relation to Bernoulli numbers

The Bernoulli numbers B_m, $m = 0, 1, \ldots,$ can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}, \quad |t| < 2\pi \tag{26}$$

[**1, 5, 9**]. From this

$$B_m = \left(\frac{d}{dt} \right)^m \frac{t}{e^t - 1} \bigg|_{t=0}. \tag{27}$$

It is tempting to evaluate these derivatives at zero by using the Leibnitz rule for the product $t \cdot \frac{1}{e^t - 1}$ and formula (22) with $\mu = -1$, $\lambda = 1$. This will not work, though, because the denominator $\mu + 1$ on the right-hand side becomes zero. To find a relation between the Bernoulli and Stirling numbers we shall use a simple trick and the generating function (18). Writing $t = \ln e^t = \ln(1 + (e^t - 1))$ we have for t small enough

$$\begin{aligned} \frac{t}{e^t - 1} &= \frac{\ln(1 + (e^t - 1))}{e^t - 1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1} (e^t - 1)^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left\{ n! \sum_{m=n}^{\infty} S(m,n) \frac{t^m}{m!} \right\}$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} \left\{ \sum_{n=0}^{m} (-1)^n \frac{n!}{n+1} S(m,n) \right\}.$$

Comparing this to (26) we find for $m = 0, 1, \ldots,$

$$B_m = \sum_{n=0}^{m} (-1)^n \frac{n!}{n+1} S(m,n). \tag{28}$$

Sums of powers

The Bernoulli numbers historically appeared in the works of the Swiss mathematician Jacob Bernoulli (1654–1705) who evaluated sums of powers of consecutive integers [**1, 9**]

$$1^m + 2^m + \cdots + (n-1)^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k},$$

for any $m \geq 0$, $n \geq 1$. This is the famous Bernoulli formula. It is interesting to see that sums of powers can also be evaluated directly in terms of Stirling numbers of the second kind. In order to do this, we invert the representation (4); i.e.,

$$n^m = \sum_{k=0}^{n} \binom{n}{k} S(m,k) k!. \tag{29}$$

This inversion is a property of the binomial transform [**15**]. Given a sequence $\{a_k\}$, its *binomial transform* $\{b_k\}$ is the sequence defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k, \tag{30}$$

and the inversion formula is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k.$$

Next, from (29),

$$1^m + 2^m + \cdots + n^m = \sum_{p=1}^{n} \left\{ \sum_{k=0}^{p} \binom{p}{k} S(m,k) k! \right\}$$

$$= \sum_{k=0}^{n} S(m,k) k! \left\{ \sum_{p=k}^{n} \binom{p}{k} \right\},$$

by changing the order of summation. Now using the well-known identity

$$\sum_{p=k}^{n} \binom{p}{k} = \binom{n+1}{k+1},$$

we finally obtain

$$1^m + 2^m + \cdots + n^m = \sum_{k=0}^{n} \binom{n+1}{k+1} S(m,k)k!,$$

which is the desired representation.

Stirling numbers of the first kind

Inverting equation (8) we have

$$z(z-1)\cdots(z-m+1) = \sum_{k=0}^{m} s(m,k)z^k$$

where the coefficients $s(m,k)$ are called *Stirling numbers of the first kind.* The following inversion property is true.

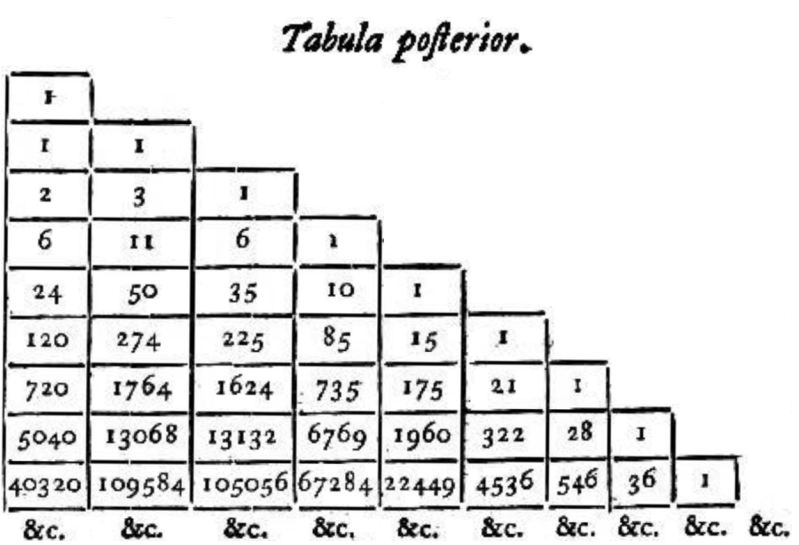
$$\sum_{k=0}^{\infty} S(m,k)s(k,n) = \delta_{mn} = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases}$$

The coefficients here come from the representation ($m = 1, 2, \ldots$)

$$\frac{1}{z^{m+1}} = \sum_{k=0}^{\infty} \frac{\sigma(m+k,m)}{z(z+1)\cdots(z+m+k)},$$

following the columns of the table. The numbers $\sigma(m,k)$ are called today the Stirling cycle numbers or the unsigned Stirling numbers of the first kind [**5, 9**]. An often-used notation is

$$\sigma(m,k) = \begin{bmatrix} m \\ k \end{bmatrix}.$$



Tabula posterior.

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6769	1960	322	28	1		
40320	109584	105056	67284	22449	4536	546	36	1	
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

Figure 4 Stirling's Second Table

We have

$$s(m,k) = (-1)^{m-k}\sigma(m,k).$$

Stirling's book [**18, 19**] contains a second table showing the values of $\sigma(m,k)$; see FIGURE 4. More properties, combinatorial interpretation, details, and generating functions can be found in the excellent books [**5, 9, 11**].

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Summary This is a short introduction to the theory of Stirling numbers of the second kind $S(m,k)$ from the point of view of analysis. It is written as an historical survey centered on the representation of these numbers by a certain binomial transform formula. We tell the story of their birth in the book *Methodus Differentialis* (1730) by James Stirling, and show how they mature in the works of Johann Grünert. The paper demonstrates the usefulness of these numbers in analysis. In particular, they appear in several differentiation and summation formulas. The reader can also see the connection of $S(m,k)$ to Bernoulli numbers, to Euler polynomials, and to power sums.

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