How to Win at (One-Round) War Supplementary Online Materials II: The Hungarian Algorithm

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This document is the second of two providing supplementary materials for the article "How to Win at (One-Round) War." The first document includes proofs omitted from the main article. It logically precedes this document, i.e., we will use some of the results proved in the first document in this document as well. This second document describes how the Hungarian algorithm provides an alternative approach to identifying the optimal strategy. It assumes some familiarity with linear algebra, linear programming and optimization problems.

This document is divided into two main sections. In the first section we describe a generic iterative approach to solving the dual problem, thereby identifying the optimal strategy. The second section demonstrates how a modified version of the Hungarian algorithm represents a specific instance of the iterative approach described in the first section, and hence how the Hungarian algorithm can be employed to establish the form of the optimal strategy.

Dual problem

A general linear assignment problem may be solved by the Hungarian algorithm. An alternative formulation of a linear assignment problem is to select a complete matching with minimum cost in the complete bipartite graph $G = (L, R, E)$, where $L = R = \{1, 2, \ldots, n\}$ correspond to the cards of Players 1 and 2, respectively, and the cost of selecting edge (i, j) with $i \in L$ and $j \in R$ is P_{ij}. The Hungarian algorithm is frequently described in this graphical context.

The linear assignment problem is a special case of a linear program, stated as the primal problem on the left below. (We do not need to include the integer constraints in the primal problem because its form guarantees an integral solution.) Each primal problem has a corresponding dual problem; the dual for our primal is given on the right.

$$\begin{array}{rl} \text{minimize} & \sum_{i,j} P_{ij} x_{ij} \\ \text{subject to} & \\ \forall i, & \sum_j x_{ij} = 1 \\ \forall j, & \sum_i x_{ij} = 1 \\ \forall i, j, & x_{ij} \geq 0. \end{array} \qquad \begin{array}{rl} \text{maximize} & \sum_i u_i + \sum_j v_j \\ \text{subject to} & \\ \forall i, j, & u_i + v_j \leq P_{ij}. \\ & \\ & \end{array}$$

The primal is a minimization problem while the dual is a maximization problem. The two are related in a number of ways. Weak duality says that any feasible solution to the primal will have an objective value that is greater than the objective value of any feasible solution to the dual. Strong duality says that any optimal solution to the primal will have the same objective value as any optimal solution to the dual.

For any dual solution $(\boldsymbol{u}, \boldsymbol{v})$, the corresponding reduced cost matrix $\boldsymbol{c}$ has (i, j)th element $c_{ij} = P_{ij} - u_i - v_j$. Dual feasibility requires that all entries of the reduced cost matrix be nonnegative. Given an optimal dual solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$, strong duality says that the revised assignment problem with costs given by the reduced cost matrix $\boldsymbol{c}^*$, with (i,j)th element $c^*_{ij} = P_{ij} - u^*_i - v^*_j$ has optimal value 0 and its optimal solutions coincide precisely with those of the original primal. The so-called complementary slackness condition follows immediately, namely that at optimality $c^*_{ij}x^*_{ij} = 0$, i.e., a non-zero primal variable has a corresponding reduced cost that is zero.

The Hungarian algorithm is a primal-dual algorithm that maintains a feasible solution to the dual and a partial solution (a partial matching in the complete bipartite graph) to the primal. At each iteration of the algorithm the partial primal solution is a partial matching that is a maximum matching in the subgraph defined by the edges with zero reduced cost. At each iteration the dual objective value increases and the size of the partial matching does not decrease. The algorithm terminates when a complete matching is achieved.

In this section we demonstrate how an iterative dual algorithm arises naturally when considering reduced cost matrices that are dual feasible. This algorithm leads naturally to establishing the strategy that sacrifices the worst $k^*(n)$ cards as being optimal. In the following section, we show how an alternative choice of the iterative steps within this algorithm corresponds to a modified version of the Hungarian algorithm.

Lemma SM14. ***(i)*** *There exists a symmetric optimal dual solution.* ***(ii)*** *([**4**], Corollary 5.10) There exists an optimal primal solution that consists only of cycles of length 2 or of odd length.*

Proof. (i) Because the cost matrix is symmetric, for any feasible dual solution $(\boldsymbol{u}, \boldsymbol{v})$, both $(\boldsymbol{v}, \boldsymbol{u})$ and $((\boldsymbol{u}+\boldsymbol{v})/2, (\boldsymbol{u}+\boldsymbol{v})/2)$ are feasible dual solutions with the same objective value. (ii) Because the cost matrix is symmetric and hence there exists a symmetric optimal dual solution by part (i), the corresponding reduced cost matrix is symmetric. Therefore, any even length cycle $(i_1, i_2, \ldots, i_{2k-1}, i_{2k})$ in the cyclic representation of an optimal primal solution can be replaced by the 2-cycles $(i_{2h-1}, i_{2h}), h = 1, 2, \ldots, k$ to produce a new feasible solution with the same objective value: $c^*_{2h-1,2h} = 0$ because $(2h-1, 2h)$ is part of the original cycle, and then $c^*_{2h,2h-1} = 0$ by symmetry of the reduced cost matrix. ■

The reduced cost matrix associated with a symmetric dual solution inherits all the properties of the original cost matrix detailed in Lemma 1 of the main article.

If the strategy that sacrifices the worst $k^*(n)$ cards is optimal, then complementary slackness tells us the optimal dual solution has a reduced cost matrix with zeros precisely corresponding to the primal solution, i.e., on the $m^*(n)$ anti-diagonal and on the main diagonal from $(m^*(n)+1, m^*(n)+1)$ to (n, n). If we can find such a reduced cost matrix with the remaining elements nonnegative, this would establish the optimality of the strategy. The following lemma suggests how we might narrow the search for such a matrix; the proof relies only on the properties detailed in Lemma 1.

Lemma SM15. *Suppose $m < n$ and let $\mathbf{c}$ be a symmetric reduced cost matrix such that all elements*

1. *on the m-anti-diagonal are zero;*
2. *on the $(m-1)$- and $(m+1)$-anti-diagonals are nonnegative;*
3. *c_{1k} $(= c_{k1})$ in the first row (column), with $m < k \leq n$, are nonnegative;*
4. *on the main diagonal below the main anti-diagonal are nonnegative.*

Then all other elements of $\mathbf{c}$ are positive.

Proof. For all elements on or above the m-anti-diagonal, we can use reverse induction on the k-anti-diagonal, where the inductive hypothesis is that all elements of the k-

anti-diagonal are nonnegative. This holds for $k = m - 1$ by condition 2; suppose it holds for some $k \le m - 1$. Consider any element of the $(k-1)$-anti-diagonal, e.g., element $(i, k-i)$. Then by supermodularity above the main anti-diagonal:

$$c_{i,k-i} + c_{i+1,m-i} > c_{i+1,k-i} + c_{i,m-i} \quad \Rightarrow c_{i,k-i} > 0,$$

because $c_{i+1,m-i} = 0$ by condition 1, $c_{i,m-i} \ge 0$ by condition 2, and $c_{i+1,k-i} \ge 0$ by the inductive hypothesis. An entirely similar argument in the forward direction demonstrates the claim for the remaining elements on or above the main-anti-diagonal that are not in the first row or column.

Finally, consider any element (i, j), with $i \ne j$, that lies below the main anti-diagonal, i.e., $i + j \ge n + 2$. Then by symmetry and submodularity below the main anti-diagonal:

$$c_{ij} = \frac{1}{2}(c_{ij} + c_{ji}) > \frac{1}{2}(c_{ii} + c_{jj}) \ge 0,$$

where the last inequality follows from condition 4. ∎

Given m, let us try to construct $\boldsymbol{u}^m$, a set of dual variables satisfying the conditions of Lemma SM15. We must choose $u_1^m, \ldots, u_m^m$ so that

$$\begin{aligned} c^m_{m-j,j} &= P_{m-j,j} - u^m_{m-j} - u^m_j \ge 0, &&\text{for } 1 \le j \le m-1 &(1)\\ c^m_{m+1-j,j} &= P_{m+1-j,j} - u^m_{m+1-j} - u^m_j = 0, &&\text{for } 1 \le j \le m &(2)\\ c^m_{m+2-j,j} &= P_{m+2-j,j} - u^m_{m+2-j} - u^m_j \ge 0, &&\text{for } 1 \le j \le m, &(3)\end{aligned}$$

and given these, for $m + 1 \le j \le n$, set

$$u_j^m \le \min\left\{\frac{1}{2}P_{jj}, P_{1j} - u_1^m\right\}. \tag{4}$$

This ensures that, for each such j, both c^m_{jj} and c^m_{1j} are nonnegative. Any such $\boldsymbol{u}^m$ satisfies conditions 1, 2, and 3 of Lemma SM15. Condition 4 will be satsified provided the elements on the main diagonal from $((n+1)/2, (n+1)/2)$ to (m, m) are nonnegative. We will address this shortly.

Focus for now on inequality (1) and equality (2). Subtracting (1) from (2) results in

$$u_j^m \le u_{j+1}^m - a_{m-j,j+1}, \quad \text{for } 1 \le j \le m-1.$$

If $m = 2k - 1$ is odd, then $u_k^m = P_{kk}/2$, else $m = 2k$ is even and $u_k^m \le P_{kk}/2$ and $u_{k+1}^m \ge P_{kk}/2 + a_{k,k+1}$. Then

$$u_j^m \le \frac{1}{2}P_{kk} - \sum_{i=j}^{k-1} a_{m-i,i+1}, \quad \text{for } 1 \le j \le k$$

$$u_j^m \ge \frac{1}{2}P_{kk} + \sum_{i=k+1}^{j} a_{m+1-i,i}, \quad \text{for } k+1 \le j \le m.$$

In other words, this gives an upper bound on the first k of the elements $u_1^m, \ldots, u_m^m$ (and a complementary lower bound on the remaining $m - k$ of those elements). These bounds are tight precisely when the inequalities in (1) are tight, i.e., all the elements of the $(m-1)$-anti-diagonal are zero.

Similarly, we can focus on equality (2) and inequality (3). Subtracting (2) from (3) results in

$$u_j^m \geq u_{j+1}^m - a_{m+1-j,j+1}, \quad \text{for } 1 \leq j \leq m-1.$$

Again, if $m = 2k+1$ is odd, then $u_{k+1}^m = P_{k+1,k+1}/2$, else $m = 2k$ is even and $u_{k+1}^m \leq P_{k+1,k+1}/2$ and $u_k^m \geq P_{k+1,k+1}/2 - a_{k+1,k+1}$. Then

$$u_j^m \geq \frac{1}{2}P_{k+1,k+1} - \sum_{i=j}^{k} a_{m+1-i,i+1}, \quad \text{for } 1 \leq j \leq k$$

$$u_j^m \leq \frac{1}{2}P_{k+1,k+1} + \sum_{i=k+2}^{j} a_{m+2-i,i}, \quad \text{for } k+1 \leq j \leq m.$$

In other words, this gives a lower bound on the first k of the elements $u_1^m, \ldots, u_m^m$ (and a complementary upper bound on the remaining $m-k$ of those elements). These bounds are tight precisely when the inequalities in (3) are tight, i.e., all the elements of the $(m+1)$-anti-diagonal are zero.

Given values of $u_1^m, \ldots, u_m^m$ satisfying (1)-(3), summing equation (2) over $1 \leq j \leq \lfloor (m+1)/2 \rfloor$ gives that $2\sum_{j=1}^{m} u_j^m = S_m$. We can maximize the objective value of the dual by demanding equality in (4). Let D^m be the set of all solutions of (1)-(4), with equality in (4). Among all members of D^m, the corresponding dual objective value is a decreasing function of u_1^m. Thus, the member $\boldsymbol{u}^{m-}$ of D^m with the smallest dual objective value is that which achieves equality in (1), i.e., for which all elements of both the $(m-1)$- and m-anti-diagonals of the corresponding reduced cost matrix $\boldsymbol{c}^{m-}$ are zero. The member $\boldsymbol{u}^{m+}$ of D^m with the largest dual objective value is that which achieves equality in (3), i.e., for which all elements of both the m- and $(m+1)$-anti-diagonals of the corresponding reduced cost matrix $\boldsymbol{c}^{m+}$ are zero; this member of D^m is also the member of D^{m+1} with the smallest dual objective value, i.e., $\boldsymbol{u}^{m+} = \boldsymbol{u}^{(m+1)-}$.

We can iterate through increasing values of m, starting with $m = 1$, at each iteration selecting the member of D^m with largest dual objective value. Then the dual objective value will increase at each iteration. Because $(n+1)/2 > m/2$, the dual variables $u_{\lceil (n+1)/2 \rceil}, \ldots, u_m$ will always be among the second half of the variables $u_1, \ldots, u_m$ and hence their values will increase from one iteration to the next. Similarly, the value of u_1 will always decrease and so the values of $u_{m+1}, \ldots, u_n$ will increase or remain unchanged. Therefore, the reduced costs $c_{\lceil (n+1)/2 \rceil, \lceil (n+1)/2 \rceil}, \ldots, c_{nn}$ will decrease or remain zero. Therefore, if they are nonnegative at one iteration, then they must have been nonnegative at all previous iterations.

We can demonstrate that $\boldsymbol{u}^{m^*(n)-}$, the member of $D^{m^*(n)}$ with the smallest dual objective value, i.e., that with all elements of both the $(m^*(n)-1)$- and $m^*(n)$-anti-diagonals of the corresponding reduced cost matrix $\boldsymbol{c}^{m^*(n)-}$ zero, has all elements of the main diagonal nonnegative. Then this must also be true of any $\boldsymbol{u}^m$ for all $m < m^*(n)$, and hence by Lemma SM15 each such $\boldsymbol{u}^m$ for all $m < m^*(n)$, and $\boldsymbol{u}^{m^*(n)-}$ represents a feasible solution for the dual.

When we have equality in (1), summing over $1 \le j \le \lfloor m/2 \rfloor$ gives that

$$2\sum_{j=1}^{m-1} u_j^{m-} = S_{m-1}, \qquad \text{whence} \qquad 2u_m^{m-} = \Delta S_m.$$

It follows that $c_{mm}^{m-} = P_{mm} - \Delta S_m$. This is positive when $m = m^*(n)$.

Given $0 \le \lambda \le 1$, set $\boldsymbol{u}^{\lambda m} = (1-\lambda)\boldsymbol{u}^{m-} + \lambda \boldsymbol{u}^{m+}$, so $\boldsymbol{c}^{\lambda m} = (1-\lambda)\boldsymbol{c}^{m-} + \lambda \boldsymbol{c}^{m+}$. Then for any such λ, $\boldsymbol{c}^{\lambda m}$ satisfies conditions 1, 2 and 3 of Lemma SM15.

On $2 \le j \le m$

$$\begin{aligned} c_{jj}^{m-} - c_{j-1,j-1}^{m-} &= (P_{jj} - 2u_j^{m-}) - (P_{j-1,j-1} - 2u_{j-1}^{m-}) \\ &= P_{jj} - P_{j-1,j-1} - 2(u_j^{m-} - u_{j-1}^{m-}) \\ &= P_{jj} - P_{j-1,j-1} - 2a_{m+1-j,j} \equiv C_{m+1-j,j}, \end{aligned}$$

and

$$\begin{aligned} c_{jj}^{\lambda m} - c_{j-1,j-1}^{\lambda m} &= (1-\lambda)(c_{jj}^{m-} - c_{j-1,j-1}^{m-}) + \lambda(c_{jj}^{m+} - c_{j-1,j-1}^{m+}) \\ &= (1-\lambda)C_{m+1-j,j} + \lambda C_{m+2-j,j}. \end{aligned}$$

Now we show, for $m \ge m^*(n)$ and any $0 \le \lambda \le 1$, that the elements of the main diagonal of the reduced cost matrix $\boldsymbol{c}^{\lambda m}$ from $\lceil (n+1)/2 \rceil$ to m form a quasi-concave sequence. We do this by looking at the difference between adjacent elements of the main diagonal. Write $\Delta_j^{\lambda m} = c_{jj}^{\lambda m} - c_{j-1,j-1}^{\lambda m}$. We will show that if $\Delta_j^{\lambda m} > 0$, then so too $\Delta_{j-1}^{\lambda m} > 0$.

Lemma SM16. *For $n \ge 3$, $m \ge m^*(n)$ and $0 \le \lambda \le 1$, the diagonal of the reduced cost matrix $\boldsymbol{c}^{\lambda m}$ on $\lfloor (m+1)/2 \rfloor \le j \le m+1$ is quasi-concave.*

Proof. Write

$$\begin{aligned} \Delta_j^{m-} &= c_{jj}^{m-} - c_{j-1,j-1}^{m-} \\ &= (P_{jj} - 2u_j^{m-}) - (P_{j-1,j-1} - 2u_{j-1}^{m-}) \\ &= P_{jj} - P_{j-1,j-1} - 2(u_j^{m-} - u_{j-1}^{m-}) \\ &= P_{jj} - P_{j-1,j-1} - 2(P_{m+1-j,j} - P_{m+1-j,j-1}) \\ &= (P_{jj} + P_{m+1-j,j-1} - P_{j,j-1} - P_{m+1-j,j}) \\ &\qquad + (P_{j-1,j} + P_{m+1-j,j-1} - P_{j-1,j-1} - P_{m+1-j,j}). \end{aligned}$$

Supermodularity above the main diagonal applies to both terms provided $m < 2j - 1 \le n+1 \Rightarrow \lfloor (m+1)/2 \rfloor < j \le \lfloor n/2 \rfloor + 1$: then $\Delta_j^{m-} > 0$. This argument holds for any value of m so we likewise have $\Delta_j^{m+} > 0$ and hence also $\Delta_j^{\lambda m} > 0$ for $\lfloor (m+1)/2 \rfloor < j \le \lfloor n/2 \rfloor + 1$.

By Lemma SM10(iii) from the first Supplementary Online Materials document, on $j \ge n + 1 - \lfloor n/4 \rfloor$, or by Lemma SM10(iv), on $j > m^*(n)$, both $C_{m+1-j,j} < 0$ and $C_{m+2-j,j} < 0$, so $\Delta_j^{\lambda m} = c_{jj}^{\lambda m} - c_{j-1,j-1}^{\lambda m} = (1-\lambda)C_{m+1-j,j} + \lambda C_{m+2-j,j} < 0$.

It suffices to show that if $\Delta_j^{\lambda m} > 0$, then also $\Delta_{j-1}^{\lambda m} > 0$. The former condition can only hold if $j < n + 1 - \lfloor n/4 \rfloor$ and $j \le m^*(n)$. The latter is guaranteed if $j - 1 \le \lfloor n/2 \rfloor + 1$, so we need only consider $j > \lfloor n/2 \rfloor + 2 \ge \lceil n/2 \rceil + 1$.

So consider $\lceil n/2 \rceil + 2 \le j \le \min\{n - \lfloor n/4 \rfloor, m^*(n)\}$, and suppose $\Delta_j^{\lambda m} = (1-\lambda)C_{m+1-j,j} + \lambda C_{m+2-j,j} > 0$. With $i = n+1-j$ and $k = n-m$, this implies that $1 - 2((1-\lambda)D_{ik} + \lambda D_{i,k-1}) > 0$. Further, $\max\{\lfloor n/4 \rfloor, k^*(n)\} < i \le \lfloor n/2 \rfloor - 1$ and $k \le k^*(n)$, so by Lemma SM12, we have $D_{ik} \ge D_{i+1,k}$ and $D_{i,k-1} \ge D_{i+1,k-1}$, whence $(1-\lambda)D_{ik} + \lambda D_{i,k-1} \ge (1-\lambda)D_{i+1,k} + \lambda D_{i+1,k-1}$. Then

$$\begin{aligned}
\Delta_{j-1}^{\lambda m} &= (1-\lambda)C_{m+2-j,j-1} + \lambda C_{m+3-j,j-1} \\
&= (1-\lambda)\hat{C}_{i+1,k} + \lambda \hat{C}_{i+1,k-1} \\
&= \binom{n}{i+1}^2 [1 - 2((1-\lambda)D_{i+1,k} + \lambda D_{i+1,k-1})] \\
&\ge \binom{n}{i+1}^2 [1 - 2((1-\lambda)D_{ik} + \lambda D_{i,k-1})] \\
&> \binom{n}{i}^2 [1 - 2((1-\lambda)D_{ik} + \lambda D_{i,k-1})] \\
&= (1-\lambda)\hat{C}_{ik} + \lambda \hat{C}_{i,k-1} \\
&= (1-\lambda)C_{m+1-j,j} + \lambda C_{m+2-j,j} = \Delta_j^{\lambda m} > 0,
\end{aligned}$$

where the last inequality holds because $1 - 2((1-\lambda)D_{ik} + \lambda D_{i,k-1}) > 0$ and $\binom{n}{i+1} > \binom{n}{i}$. ■

Lemma SM17. *For $n \ge 3$ and $m < n$, for any $\boldsymbol{u}^m \in D^m$, the diagonal of the corresponding reduced cost matrix $\mathbf{c}^m$ is nonincreasing on $\max\{m, m^*(n)\} < j \le n$. If $m \le m^*(n)$, then the diagonal of $\mathbf{c}^m$ is nonincreasing on $m^*(n) \le j \le n$.*

Proof. We need to demonstrate that $\Delta_j^m \le 0$ for $\max\{m, m^*(n)\} + 1 < j \le n$. Recall that on this range,

$$u_j^m = \min\left\{\frac{1}{2}P_{jj}, P_{1j} - u_1^m\right\} \text{ and } u_{j-1}^m = \min\left\{\frac{1}{2}P_{j-1,j-1}, P_{1,j-1} - u_1^m\right\},$$

so that $c_{jj}^m \ge 0$ and $c_{j-1,j-1}^m \ge 0$. Clearly, if $c_{jj}^m = 0$, then $\Delta_j^m \le 0$. If $c_{jj}^m > 0$, then $u_j^m = P_{1j} - u_1^m$ and $c_{jj}^m = P_{jj} + 2u_1^m - 2P_{1j}$. Now $u_{j-1}^m \le P_{1,j-1} - u_1^m$ so $c_{j-1,j-1}^m \ge P_{j-1,j-1} + 2u_1^m - 2P_{1,j-1}$, whence $\Delta_j^m \le P_{jj} - P_{j-1,j-1} - 2a_{1j} = C_{1j} < 0$ by Lemma SM10(iv) because $j > m^*(n)$.

The same argument applies when $j = m^*(n) + 1$ provided $u_{m^*(n)}^m \le P_{1,m^*(n)} - u_1^m$. This is certainly true when $m < m^*(n)$. When $m = m^*(n)$, $c_{mm}^m \ge 0$ by definition, whence $u_m^m = P_{1m} - u_1^m$. ■

Now consider the specific case in which $m \equiv m^*(n)$. If $c_{m+1,m+1}^{m-} = 0$, set $\lambda^* = 0$, else we have $c_{m+1,m+1}^{m-} > 0 \ge c_{m+1,m+1}^{m+}$, so we can set

$$\lambda^* = \frac{c_{m+1,m+1}^{m-}}{c_{m+1,m+1}^{m-} - c_{m+1,m+1}^{m+}},$$

so that $0 \le \lambda^* \le 1$ and $c_{m+1,m+1}^{\lambda^* m} = 0$. Then we must have that $u_{m+1}^{\lambda^* m} = P_{m+1,m+1}/2$.

Lemma SM18. *For $n \geq 3$, $m = m^*(n)$, and $0 \leq \lambda \leq \lambda^*$, the diagonal of the reduced cost matrix $\boldsymbol{c}^{\lambda m}$ on $\lfloor (m+1)/2 \rfloor \leq j \leq n$ is quasi-concave and nonnegative.*

Proof. The previous two lemmas establish quasi-concavity on the range $\lfloor (m+1)/2 \rfloor \leq j \leq n$ because $c^{\lambda m}_{mm} \geq 0$ by the definitions of $m^*(n)$ and λ^*. Then the minimum element on this range must be at one of the endpoints. But $c^{\lambda m}_{\lfloor (m+1)/2 \rfloor, \lfloor (m+1)/2 \rfloor} \geq 0$ and $c^{\lambda m}_{nn} = 0$, so all the elements must be nonnegative. ∎

Corollary SM19. *For $n \geq 3$ and $m < m^*(n)$, any member of D^m is feasible for the dual.*

Proof. By the previous lemma and Lemma SM15, $\boldsymbol{u}^{m^*(n)-}$ is feasible for the dual. The elements of the diagonal of the reduced cost matrix below the main anti-diagonal for any member of D^m exceed those of $\boldsymbol{c}^{m^*(n)-}$ and hence are nonnegative. The conclusion follows from Lemma SM15. ∎

Define the primal solution $\boldsymbol{x}^*$ to be the complete matching consisting of all elements on the $m^*(n)$-anti-diagonal together with elements $(m^*(n)+1, m^*(n)+1), \ldots, (n, n)$. Define $\boldsymbol{u}^* = \boldsymbol{u}^{\lambda^* m^*(n)}$.

Theorem SM20. *The solutions $\boldsymbol{x}^*$ and $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}^*, \boldsymbol{u}^*)$ are optimal for the primal and dual, respectively.*

Proof. The primal solution $\boldsymbol{x}^*$ is clearly feasible. Similarly, $(\boldsymbol{u}^*, \boldsymbol{u}^*)$ is feasible for the dual as all the conditions of Lemma SM15 are satisfied for the corresponding reduced cost matrix $\boldsymbol{c}^*$ (Lemma SM18 demonstrates that condition 4 is satisfied). Further, $c^*_{m^*(n)+1, m^*(n)+1} = 0$, and so by Lemma SM17 $c^*_{jj} = 0$ for all $m^*(n)+1 \leq j \leq n$. Then the complementary slackness conditions are satisfied: $c^*_{ij} x^*_{ij} = 0$ for all i and j. We conclude that both $\boldsymbol{x}^*$ and $(\boldsymbol{u}^*, \boldsymbol{u}^*)$ are optimal. ∎

Solution via a modified-Hungarian algorithm

Here we show how a modified version of the Hungarian algorithm corresponds to starting with dual solution $\boldsymbol{u}^{1+} = \boldsymbol{u}^{2-}$, then for each $2 \leq m < m^*(n)$, iterating through members of D^m from $\boldsymbol{u}^{m-}$ to $\boldsymbol{u}^{m+}$, and finally iterating through members of $D^{m^*(n)}$ from $\boldsymbol{u}^{m^*(n)-}$ until reaching an optimal dual solution.

The first step in the Hungarian algorithm is to subtract the minimum element in each row from all elements in the row and then the minimum element in each column from all the elements in that column. Given the form of our cost matrix, the minimum element in each row will be found in the first column and the minimum element in each column will be found in the first row. In the 7-card example, this results in the following reduced cost matrix.

	1	2	3	4	5	6	7	u_i
1	0	0	0	0	0	0	0	0.5
2	0	35	133	329	623	917	917	7.5
3	0	133	420	861	1351	1645	1351	35.5
4	0	329	861	1477	1967	2065	1477	119.5
5	0	623	1351	1967	2282	2135	1351	329.5
6	0	917	1645	2065	2135	1799	917	791.5
7	0	917	1351	1477	1351	917	0	1715.5
v_j	0.5	7.5	35.5	119.5	329.5	791.5	1715.5	

This is obtained by setting the dual variables to

$$u_i = v_i = P_{1i} - \frac{1}{2}P_{11},$$

which results in the reduced costs

$$c_{ij} = P_{ij} - P_{i1} - P_{1j} + P_{11}.$$

We recognize this first step in the Hungarian algorithm as $\boldsymbol{u}^{1+} = \boldsymbol{u}^{2-}$. The initial partial matching for the primal problem has size 3, obtained by setting x_{21}, x_{12}, and x_{nn} equal to 1.

Modified Pivoting for the Hungarian algorithm Each step of the Hungarian algorithm involves a pivot on an element of the reduced cost matrix equivalent to a pivot in the Simplex algorithm. To find the element on which to pivot, we first cover some number of rows or columns of the reduced cost matrix so that all zeros of the matrix are covered. The minimum number of rows or columns of the reduced cost matrix that must be covered so that all zeros of the matrix are covered is equal to the size of the current partial matching. Typically in the Hungarian algorithm one covers the minimum number of rows or columns required because this leads to the largest increase in the dual objective, and hence (in a greedy sense) may minimize the number of iterations of the algorithm to reach optimality. This is the policy we will follow in our modified algorithm.

	~~1~~	2	3	4	5	6	7
~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
2	~~0~~	35	133	329	623	917	917
3	~~0~~	133	420	861	1351	1645	1351
4	~~0~~	329	861	1477	1967	2065	1477
5	~~0~~	623	1351	1967	2282	2135	1351
6	~~0~~	917	1645	2065	2135	1799	917
~~7~~	~~0~~	~~917~~	~~1351~~	~~1477~~	~~1351~~	~~917~~	~~0~~

We pivot on the least uncovered element, in this case the $(2, 2)$ element for which $c_{22} = 35$. The standard pivot in the Hungarian algorithm results in all of the uncovered elements in the reduced cost matrix decreasing by the value of the pivot element (so that the reduced cost of the pivot element becomes 0), all reduced costs of elements covered only once remaining unchanged, and the reduced costs of any elements covered twice increasing by the value of the pivot element. In our example, this could be achieved by adjusting the dual variables in a variety of ways. However, in order to maintain symmetry of the dual solution, we decrease u_1 and v_1 by 17.5, increase u_2 through u_6 and v_2 through v_6 by 17.5 and leave u_7 and v_7 unchanged. While the standard algorithm allows many alternative adjustments to the dual variables when pivoting on a diagonal element, our requirement that the dual solution remain symmetric at all times means modifying the standard Hungarian algorithm to admit a unique adjustment to the dual variables. The revised reduced cost matrix and dual values are as follows.

	1	2	3	4	5	6	7	u_i
1	35	0	0	0	0	0	17.5	−17.0
2	0	0	98	294	588	882	899.5	25.0
3	0	98	385	826	1316	1610	1333.5	53.0
4	0	294	826	1442	1932	2030	1459.5	137.0
5	0	588	1316	1932	2247	2100	1333.5	347.0
6	0	882	1610	2030	2100	1764	899.5	809.0
7	17.5	899.5	1333.5	1459.5	1333.5	899.5	0	1715.5
v_j	−17.0	25.0	53.0	137.0	347.0	809.0	1715.5	

This solution is $\boldsymbol{u}^{2+} = \boldsymbol{u}^{3-}$. Observe that as a result of the first pivot we can now achieve a partial matching of size 4 by setting x_{31}, x_{22}, x_{13}, and x_{77} to 1. We must then cover four rows and columns to cover all the zeros, as follows. Whenever we have a choice between covering a row or covering a column, if the zero being covered is on or above the main anti-diagonal, we choose to cover the column, else we choose to cover the row.

	1	2	3	4	5	6	7	u_i
1	35	0	0	0	0	0	17.5	−17.0
2	0	0	98	294	588	882	899.5	25.0
3	0	98	385	826	1316	1610	1333.5	53.0
4	0	294	826	1442	1932	2030	1459.5	137.0
5	0	588	1316	1932	2247	2100	1333.5	347.0
6	0	882	1610	2030	2100	1764	899.5	809.0
7	17.5	899.5	1333.5	1459.5	1333.5	899.5	0	1715.5
v_j	−17.0	25.0	53.0	137.0	347.0	809.0	1715.5	

The new pivot element is element $(2, 3)$. Any pivot on an off-diagonal element (i, j) will inevitably break the symmetry of the dual solution because the pivot will force the reduced cost of element (i, j) to 0 but the symmetry element (j, i) will be unchanged. We can restore the symmetry by immediately pivoting on element (j, i). This double pivot forces both element (i, j) and its symmetry element (j, i) to 0; this requires reducing some elements of the reduced cost matrix by $c_{ij} + c_{ji} = 2c_{ij}$. In our example, we perform the double pivot with a pivot value of $\delta = 98$. The dual values change as follows: u_3, u_4, u_5, and u_6 each increase by 98, u_2 and u_7 are unchanged, and u_1 decreases by 98. The revised reduced cost matrix and dual values are as follows.

	1	2	3	4	5	6	7	u_i
1	231	98	0	0	0	0	115.5	−115.0
2	98	0	0	196	490	784	899.5	25.0
3	0	0	189	630	1120	1414	1235.5	151.0
4	0	196	630	1246	1736	1834	1361.5	235.0
5	0	490	1120	1736	2051	1904	1235.5	445.0
6	0	784	1414	1834	1904	1568	801.5	907.0
7	115.5	899.5	1235.5	1361.5	1235.5	801.5	0	1715.5
v_j	−115.0	25.0	151.0	235.0	445.0	907.0	1715.5	

This new solution is $\boldsymbol{u}^{3+} = \boldsymbol{u}^{4-}$ and admits a partial matching of size 5.

General pivots We will find that at any stage of the algorithm, the reduced cost matrix is such that all conditions of Lemma SM15 are satisfied.

By Lemma SM17 and the definition of $m^*(n)$, for any $m \leq m^*(n)$, there is a $t > m^*(n) \geq m$ such that the last $n-t$ elements on the main diagonal are zero and these are the only zeros on the main diagonal below the main anti-diagonal.

Selecting the pivot element Choose as pivot element any positive element (r, s) on the $(m+1)$-anti-diagonal with $r \leq s$.

Covering rows and columns The reduced cost matrix admits a partial matching of size $n-t+m$. We can cover all the zeros in the matrix by covering the first $r-1$ and the last $n-t$ rows, together with the first $s-1$ columns (for a total of $n-t+r+s-2 = n-t+m$ covered rows and/or columns). All elements of anti-diagonals less than $(m+1)$ are covered; all elements of the $(m+1)$-anti-diagonal except (r, s) are covered.

Setting the pivot value Given pivot element (r, s), we set the pivot value δ to $\min\{c_{rs}, c_{tt}/2\}$ when the element is not on the main diagonal, i.e., $r < s$, and to $\min\{c_{rr}/2, c_{tt}/2\}$ when the element is on the main diagonal, i.e., $r = s$. When $\delta = c_{tt}/2$, this corresponds to a pivot (or double pivot) on element (t, t) in the standard Hungarian algorithm. In our context, it is perhaps simpler to think of this as a partial pivot on element (r, s).

Performing the pivot Given a pivot value of δ:

- If both row i and column i are uncovered, increase the value of u_i and v_i by δ.
- If only one of row i and column i is covered, leave the values of u_i and v_i unchanged.
- If both row i and column i are covered, decrease the value of u_i and v_i by δ.

Pivoting on the (r, s) element, the changes to the reduced cost matrix and dual variables are as follows:

	$1 \leq j < r$	$r \leq j < s$	$s \leq j \leq t$	$t < j \leq n$	Δu_i
$1 \leq i < r$	$+2\delta$	$+\delta$	0	$+\delta$	$-\delta$
$r \leq i < s$	$+\delta$	0	$-\delta$	0	0
$s \leq i \leq t$	0	$-\delta$	-2δ	$-\delta$	$+\delta$
$t < i \leq n$	$+\delta$	0	$-\delta$	0	0
Δv_j	$-\delta$	0	$+\delta$	0	

If $r = s$, then we need fewer blocks:

	$1 \leq j < r$	$r \leq j \leq t$	$t < j \leq n$	Δu_i
$1 \leq i < r$	$+2\delta$	0	$+\delta$	$-\delta$
$r \leq i \leq t$	0	-2δ	$-\delta$	$+\delta$
$t < i \leq n$	$+\delta$	$-\delta$	0	0
Δv_j	$-\delta$	$+\delta$	0	

Demonstrating increase in the dual objective value Of the dual variables, $r-1$ decrease by δ and $t-s+1$ increase by δ, meaning the change in the dual objective is $2(t+2-r-s)\delta = 2(t-m)\delta > 0$.

Demonstrating that dual feasibility is maintained Elements on the m-anti-diagonal remain 0. Elements on the $(m-1)$-anti-diagonal either increase in value or remain unchanged, so remain non-negative. On the $(m+1)$-anti-diagonal, only the values of elements (r, s) and (s, r) are changed; these decrease to 0; thus, all such elements remain non-negative. Likewise, elements in the first row remain non-negative. Therefore, the new solution is a member of D^m.

The reduced cost of any element (j, j) on the main diagonal with $j \geq r$ does not increase. Because $r \leq (m+2)/2 \leq (n+1)/2$ (because $m \leq n-1$), this means that the reduced cost for any element of the main diagonal below the main anti-diagonal does not increase. If these do not decrease so much as to become negative, then all the conditions of Lemma SM15 are satisfied, and the reduced costs matrix is non-negative and the new dual solution is feasible. Because the new solution is a member of D^m, Corollary SM19 shows that this is indeed the case.

Completing the example Given that we can achieve a partial matching of size 5, we must cover five rows and columns to cover all the zeros, as follows.

	1	2	3	4	5	6	7	u_i
1	231	98	0	0	0	0	115.5	−115.0
2	98	0	0	196	490	784	899.5	25.0
3	0	0	189	630	1120	1414	1235.5	151.0
4	0	196	630	1246	1736	1834	1361.5	235.0
5	0	490	1120	1736	2051	1904	1235.5	445.0
6	0	784	1414	1834	1904	1568	801.5	907.0
7	115.5	899.5	1235.5	1361.5	1235.5	801.5	0	1715.5
v_j	−115.0	25.0	151.0	235.0	445.0	907.0	1715.5	

Pivoting on element $(2, 4)$ with pivot value 196 results in the new reduced cost matrix:

	1	2	3	4	5	6	7	u_i
1	623	294	196	0	0	0	311.5	−311.0
2	294	0	0	0	294	588	899.5	25.0
3	196	0	189	434	924	1218	1235.5	151.0
4	0	0	434	854	1344	1442	1165.5	431.0
5	0	294	924	1344	1659	1512	1039.5	641.0
6	0	588	1218	1442	1512	1176	605.5	1103.0
7	311.5	899.5	1235.5	1165.5	1039.5	605.5	0	1715.5
v_j	−311.0	25.0	151.0	431.0	641.0	1103.0	1715.5	

Now we pivot on element $(3, 3)$ with pivot value 94.5 to obtain the new solution, which represents $\boldsymbol{u}^{4+} = \boldsymbol{u}^{5-}$ and admits a partial matching of size 6:

	1	2	3	4	5	6	7	u_i
1	812	483	196	0	0	0	406	−405.5
2	483	189	0	0	294	588	994	−69.5
3	196	0	0	245	735	1029	1141	245.5
4	0	0	245	665	1155	1253	1071	525.5
5	0	294	735	1155	1470	1323	945	735.5
6	0	588	1029	1253	1323	987	511	1197.5
7	406	994	1141	1071	945	511	0	1715.5
v_j	−405.5	−69.5	245.5	525.5	735.5	1197.5	1715.5	

Pivoting on element $(2, 5)$ with pivot value 294 results in:

	1	2	3	4	5	6	7	u_i
1	1400	777	490	294	0	0	700	−699.5
2	777	189	0	0	0	294	994	−69.5
3	490	0	0	245	441	735	1141	245.5
4	294	0	245	665	861	959	1071	525.5
5	0	0	441	861	882	735	651	1029.5
6	0	294	735	959	735	399	217	1491.5
7	700	994	1141	1071	651	217	0	1715.5
v_j	−699.5	−69.5	245.5	525.5	1029.5	1491.5	1715.5	

Finally, a partial pivot on element $(3, 4)$ (or equivalently, a pivot on element $(6, 6)$) with pivot value 199.5 forces element $(6, 6)$ to zero and results in the optimal solution:

	1	2	3	4	5	6	7	u_i
1	1799	1176	689.5	294	0	0	899.5	−899.0
2	1176	588	199.5	0	0	294	1193.5	−269.0
3	689.5	199.5	0	45.5	241.5	535.5	1141	245.5
4	294	0	45.5	266	462	560	871.5	725.0
5	0	0	241.5	462	483	336	451.5	1229.0
6	0	294	535.5	560	336	0	17.5	1691.0
7	899.5	1193.5	1141	871.5	451.5	17.5	0	1715.5
v_j	−899.0	−269.0	245.5	725.0	1229.0	1691.0	1715.5	

Uniqueness of the optimal solution

The following result from is helpful. Given an optimal reduced cost matrix, we can permute the rows and columns to arrange that the main diagonal of the reduced cost matrix is all zero, and hence that the identity permutation $(1)(2)\ldots(n)$ is optimal. Define the graph $G_{c^*}(N, A)$ with node set $N = \{1, 2, \ldots, n\}$, and edges $(i, j) \in A$ if and only if $i \neq j$ and $c^*_{ij} = 0$. The following result is immediate from the complementary slackness conditions.

Theorem 1. *([4], Proposition 5.11) The optimal primal solution is unique if and only if the graph G_{c^*} is acyclic.*

We know that the complete matching consisting of all elements on the $m^*(n)$-anti-diagonal together with elements $(m^*(n)+1, m^*(n)+1), \ldots, (n, n)$ is optimal for the primal and that some member of $D^{m^*(n)}$ is optimal for the dual. The reduced cost matrix corresponding to this optimal dual solution has the following properties:

1. All elements on the $m^*(n)$-anti-diagonal are zero;
2. All elements on the $(m^*(n)-1)$- and $(m^*(n)+1)$-anti-diagonals are nonnegative;
3. For $1 \leq i \leq m^*(n) - 1$, at least one of the elements $(i, m^*(n) - i)$ and $(i + 1, m^*(n) + 1 - i)$ is positive;
4. For $m^*(n) + 1 \leq i \leq n$, element (i, i) is zero;
5. All other elements are positive.

It is clear that we can simply reverse the order of the first $m^*(n)$ columns in order to make the main diagonal of the matrix all zero. The adjacency matrix of the graph

$G_{c^*}(N, A)$ must then have the following form:

	1	2	3	4	5	6	7
1	0	a_1	0	0	0	1	0
2	a_2	0	b_1	0	0	0	0
3	0	b_2	0	c_1	0	0	0
4	0	0	c_2	0	d_1	0	0
5	0	0	0	d_2	0	0	0
6	0	0	0	0	1	0	0
7	0	0	0	0	0	0	0

where at most one of each pair (a_1, a_2), etc., can be 1. It is immediately clear that this graph only contains a cycle when it is of the form:

	1	2	3	4	5	6	7
1	0	0	0	0	0	1	0
2	1	0	0	0	0	0	0
3	0	1	0	0	0	0	0
4	0	0	1	0	0	0	0
5	0	0	0	1	0	0	0
6	0	0	0	0	1	0	0
7	0	0	0	0	0	0	0

i.e., the original optimal reduced cost matrix has both $m^*(n)$ and $m^*(n) + 1$ anti-diagonals zero. In this case the unique optimal dual solution is $\boldsymbol{u}^{m^*(n)+}$ and requires that

$$\begin{aligned} c^{m^*(n)+}_{m^*(n)+1,m^*(n)+1} &= P_{m^*(n)+1,m^*(n)+1} - 2a_{1,m^*(n)+1} \\ &= P_{m^*(n)+1,m^*(n)+1} - \Delta S_{m^*(n)+1} = 0. \end{aligned}$$

This is of course precisely the condition we established earlier for there to be two optimal primal solutions.

To summarize, when $P_{m^*(n)+1,m^*(n)+1} - \Delta S_{m^*(n)+1} < 0$, there is exactly one optimal primal solution and an infinite number of optimal dual solutions. Otherwise, $P_{m^*(n)+1,m^*(n)+1} - \Delta S_{m^*(n)+1} = 0$, and there are exactly two optimal primal solutions and a unique optimal dual solution.

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