Proof of Fact 1. Looking at right triangle BDE, we have $DE = (BD)\tan\alpha$, where $\alpha = \angle DBE$, or

$$x = (1-y)\tan\alpha. \tag{A}$$

From triangle ACB and the law of sines, we have

$$\sin(\theta+\alpha) = \frac{y+2r}{2r}\sin\theta, \tag{B}$$

so that $\alpha = \sin^{-1}\left(\left(1+\frac{y}{2r}\right)\sin\theta - \theta\right)$. This gives

$$x = (1-y)\tan\left(\sin^{-1}\left(\left(1+\frac{y}{2r}\right)\sin\theta - \theta\right)\right).$$

Proof of Fact 2. For this we need just the usual approximations:

$$u \approx \sin u \approx \tan u \approx \sin^{-1} u, \text{ for } u \to 0.$$

Proof of Fact 3. $\left(\frac{(1-y)y}{2r}\theta\right)$ is quadratic in y with its maximum at the average of its zeros, namely, $y = 1/2$.