Proof of Fact 7. We want to show that $f'(0) < -f'(1)$. Letting $a = 1/(2r)$, this translates into proving that $a\tan\theta < \tan((\sin^{-1}((a+1)\sin\theta) - \theta)$, or

$$\tan^{-1}(a\tan\theta) + \theta < \sin^{-1}((a+1)\sin\theta). \tag{C}$$

Note that $a > 1$ and that $(a+1)\sin\theta > 1$ (our condition of the maximum angle). Both sides of the inequality in (C) are zero for $\theta = 0$, so we are finished if the inequality holds when differentiated. That is, we are done if we can show that

$$1 + \frac{a\sec^2\theta}{1 + a^2\tan^2\theta} < \frac{(a+1)\cos\theta}{\sqrt{1-(a+1)^2\sin^2\theta}}. \tag{D}$$

Squaring both sides, cross-multiplying, then gathering everything to the right (brute-force here; I won't say if I had any electronic assistance), our inequality in (D) is true if

$$a^2\sin^2\theta(3 - (a^2+2a+3)\sin^2\theta) > 0.$$

Again using the fact that $\sin\theta < \frac{1}{a+1}$, we have

$$\begin{aligned} 3 - (a^2+2a+3)\sin^2\theta &> 3 - \frac{a^2+2a+3}{(a+1)^2} \\ &= \frac{2a(a+2)}{(a+1)^2}, \end{aligned}$$

which is positive, so we're done.