say, in a college of 1800 students, it would be infeasible to review individual preference lists. A simple one-name-per-ballot plurality procedure may be simplest and sufficient. Usually, practical common sense wins out in the end. Of course, if there is contention about which electoral procedure is best, a society can always take a vote on which one to use!

Acknowledgments and Further Reading

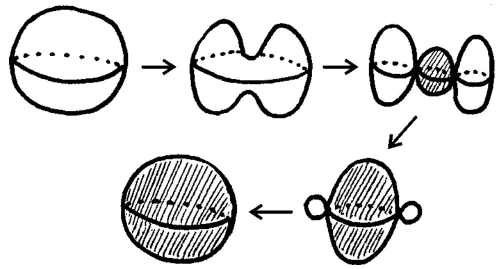
Arrow's Theorem is discussed thoroughly in Paul Hoffman's wonderful book [Hoff] and briefly by Ravi Vakil [Vaki]. An excellent elementary overview of voting theory appears in [COMA], chapter 11, which gives the real world example of the 1980 U.S. Senate race in New York where the paradox of plurality voting actually did occur. D. Saari and F. Valognes' article [Saar] is well worth a look.

The truel is a famous problem in probability theory that is still being studied today [Kilg], [Knut]. The nontransitive dice also appear in [Berl2], [Hons3], and [Vaki], and the coin game in [Vaki].

17 Laundry Math

17.1 Turning Clothes Inside Out

Taking it Further Answer. At first this question seems alarming. We could argue that a closed sphere has an interior space and an exterior space, and that the interior will always remain interior and the exterior space exterior no matter how we stretch, bend, or pull the surface. Everting the sphere would somehow mean switching the two spaces, which, quite simply, is impossible—if we stick with the current rules of the game. Imagine, however, a super high-tech material with the amazing ability to pass through itself. Self-intersection makes the eversion of a closed sphere possible. All one need do is push the north pole of the sphere down through the south and the south pole up through the north!



On one level this is a satisfactory solution, but on another it is not. Notice that a crease forms around the equator of the sphere as the sphere is pushed through itself. As we keep pulling, this crease will become sharp and the material is likely to snap! Is there a way to perform this eversion so as to avoid all sharp creases and consequent damage to the material?

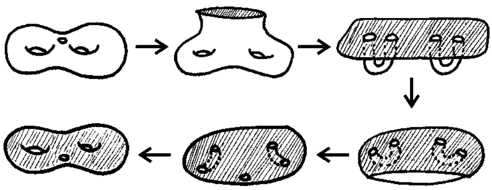
In 1957 Steve Smale [ref?] published his amazing discovery that a *smooth* eversion of a sphere is indeed possible. His celebrated work was a magnificent piece of theory but had one practical drawback: It didn't actually show how to perform the eversion. It took another seven years before Arnold Shapiro [ref?] found a practical method of eversion, but even his technique was hard to comprehend. It wasn't until 1974 when Bill Thurston [ref?] developed a scheme of "corrugations" that a comprehensible method was finally demonstrated.

The Geometry Center at the University of Minnesota has produced a computer-animated video, *Outside In* [Levy] that lets you see this eversion in all its splendor. It is a magnificent video well worth investigating. After watching you might wonder whether a smooth eversion of a non-punctured donut is possible using the same technique. (*Answer:* It is!)

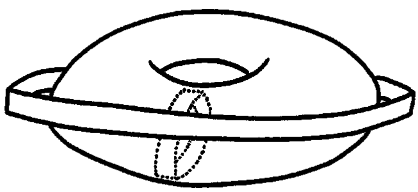
Challenge. You could also evert a sphere by making use of the fourth dimension (See "A Note on the Fourth Dimension" starting on p. 135). Can you see how?

17.2 Mutilated Laundry

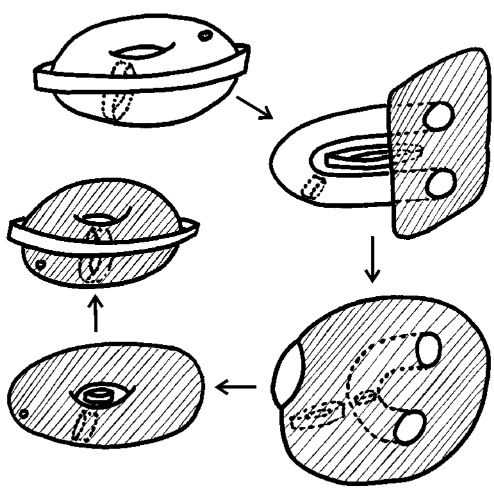
Even eversions of multi-donuts yields the same surfaces back again!

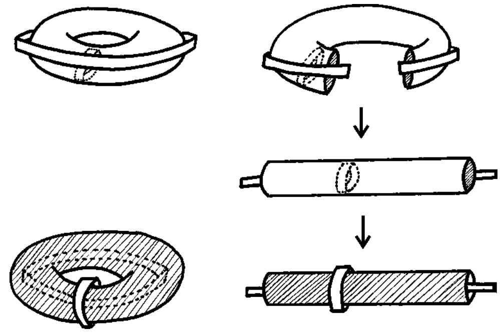


Taking it Further Answer. It turns out that the two alternative eversions of a donut we mentioned are physically distinct. No manipulation whatsoever (barring cutting and ripping) will convert one eversion into the other. To see why, imagine two rubber bands wrapped around the tube of the donut, one on the inside of the tube aligned in one direction, and the other on the outside of the torus in an orthogonal direction.



These two bands are unlinked and must remain so no matter how we manipulate the donut, even if we evert it. Our first type of eversion places the two bands into the following configuration.





The second eversion (which, for now, I know how to perform only via cutting, everting, and repasting) produces the configuration above.

The two bands are now linked! As it is impossible to link two unlinked rings in 3-space, it must be impossible to convert our first eversion of the donut into the second via physical manipulations (avoiding cutting and tearing). The two eversions are distinct.

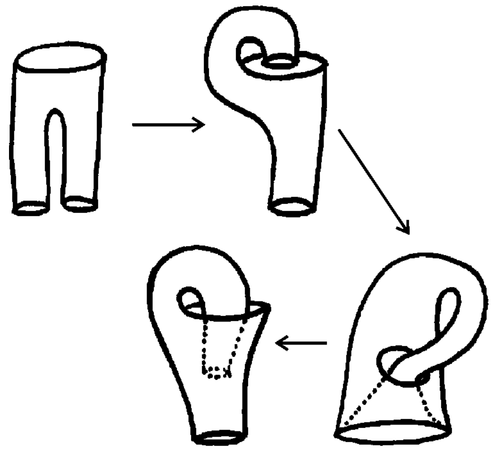
Challenge 1. Show by going to the fourth dimension that it is possible to manipulate one eversion of a punctured donut into the other (see section 10.2).

Challenge 2. How many physically distinct eversions of a punctured double-donut exist?

Challenge 3. Spheres, donuts, and multi-donuts are said to be *orientable* surfaces; they clearly have an "inside" and an "outside." However, there is another class of surfaces that I have side-stepped thus far.

Let's again start with a pair of trousers, but do not immediately sew the two leg openings together. First bring one leg up, over, and through the waist; then push it down through the tube of the second leg; now sew! The resulting shape is complicated, but it is one that the mathematical community recognizes. It is a punctured Klein bottle. (See also chapter 10.)

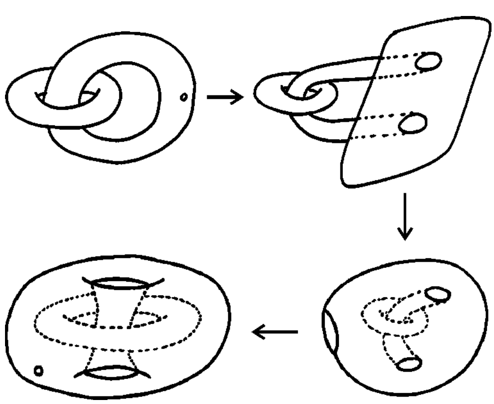
This is an example of a "one-sided" or *non-orientable* surface. An ant crawling on this shape could reach either side of the material without ever cheating by crossing over the edge of the hole. (Since we sewed the two legs together, the ant is allowed to cross over the new

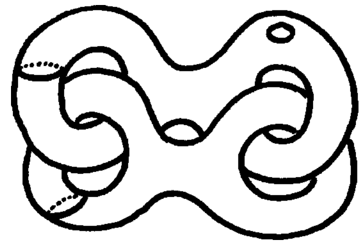


seamwork.) This surface has no “inside” or “outside” and the question of turning it inside out may be perturbing. Nonetheless, one can pull the material through itself to reveal an everted shape, and again it is exactly the same in structure. The eversion of a Klein bottle is another Klein bottle. (Try it!) The trouser material is, as expected, inside out, but this time the second leg is down inside the tube of the first.

You could also obtain an everted Klein bottle by first turning the pair of trousers inside out and then sewing the two leg openings together. Is this a new eversion of the Klein bottle, physically distinct from the first? Or is it just the first eversion in another guise?

17.3 Cannibalistic Clothing





Challenge. What about carnivorous jumpsuits? Is it possible for a hungry double-donut to consume an unlucky peer? How would the victim sit inside the victor?

Acknowledgments and Further Reading

To learn more about Arnold Shapiro’s eversion of a sphere have a look at [Fran1] and [Phil]. Martin Gardner writes about donut eversion problems in [Gard15], chapter 5. Also have a look at Herbert Taylor’s whimsical article [Tayl].

18 Get Knotted

18.1 Party Trick I: Two Linked Rings?

It is impossible to unlink two linked rings, and the puzzle appears unsolvable. However, we are not actually dealing with two linked rings: a little space exists between Jason’s wrist and the loop of string around it. To escape, Paul should push a fold of his string through this space, have Jason push his hand through this fold, and then withdraw the fold. This will do the trick!

Comment. This trick reminds me of another puzzler. Holding the ends of a long piece of string, one end in each hand, is it possible to tie a knot in the string without ever letting go? The