Preface

Introduction

Calculus: An Active Approach with Projects is a collection of materials for first-year calculus developed and tested at Ithaca College. It is not a complete textbook, but a complementary volume that can be used in conjunction with any textbook.

The authors are convinced that students who are actively involved in class are more likely to succeed than those who are passive. We also view calculus as a unified subject rather than a linearly ordered sequence of topics and believe that this view should be conveyed to students from the outset of their studies. The materials in *Calculus: An Active Approach with Projects* were designed with these beliefs in mind.

There are two main sections in this text. The first section contains activities that can be done in class or as homework. The second section contains large projects for the students to work on (usually in groups) outside the classroom.

Calculus instructors are free to make copies of any activity or project in this book for class use as long as the copyright notice appears.

Activities

The class activities are designed to accustom students to active participation in the course and to introduce some of the material and methods we have identified as important. Through the activities, students participate in the development of many of the central ideas of calculus. A key is active participation. This kind of student involvement fosters understanding and retention of course material. By doing activities, students also learn modeling and how to organize their approach to solving problems—both are useful for successful completion of projects. Activities also help students learn how to draw and interpret graphs—a key element in learning ways to represent functions that are not necessarily given as formulas. Finally, the first set of activities provides an overview of most of first-semester calculus. Doing a number of these activities early in the course helps students see the unity of the subject.

Each activity requires between five and fifty minutes to complete—our estimate for each activity in the book is given in the "Instructor Notes for Activities." Activities that you schedule to be done in class can, at your option, be done by individual students, or by pairs or groups of students working together. Many are suitable for use as homework assignments. In our courses we like to have an activity in as many classes as possible.

Projects

Projects serve to reinforce material already presented, motivate concepts, or introduce topics that might not otherwise be studied. Most projects involve more than one mathematical concept, and many have open-ended components. We have students work on the projects in teams of three or four, submitting a single report, although many of the projects have parts that are completed by individual students. Most of the projects require two to three weeks to complete. In a typical semester course, we have our students do two to four projects.

A significant benefit of this project-oriented approach is that students learn to solve non-trivial, multi-step problems. Working on the (shorter) activities in a guided classroom environment helps them succeed on projects.

A Modern Calculus Course

Efforts to revise the way calculus is taught have focused on a number of different issues. As stated above, the materials presented in *Calculus: An Active Approach with Projects* are designed to empower the student to take an active role in her or his own learning. We emphasize the role of calculus as a tool for understanding the world and hence focus on modeling as a central theme. We also emphasize the notion of function and are careful to show that functions can be represented in many different ways: as graphs, as tables of values, as algebraic expressions, as descriptions (written or verbal), as physical relationships, and as theoretical models.

Course Logistics

Curriculum

Teaching a calculus course using modern materials usually requires some modification of one's teaching style and some reorganization of topics. As an aid in this adaptation, we have included a *sample* curriculum for a two-semester sequence. The curriculum is offered as an "existence proof" rather than a prescription—there are many possible variations that will result in a successful course based on the materials in *Calculus: An Active Approach with Projects*. The sample curriculum is included as Appendix A in this book.

The sample curriculum illustrates a change in course organization—the spiral approach—that we have found to be particularly effective. Our goal is to present the main ideas of the course early so that the students will see calculus as a unified subject. The emphasis at this stage is on concepts and relationships, not on technical details.

We use the "calculus of graphs" for this purpose. That is, representing the functions involved almost exclusively in graphical form, and using the familiar ideas of velocity and distance as examples, we examine basic ideas from both differential and integral calculus. Within days, the students have some basic understanding about rates and slopes, concavity, and integration (in the context of obtaining a distance graph when given the corresponding velocity graph).

During the rest of the course, students encounter the same ideas again and again, each time picking up more of the technical and computational details.

Exams and Quizzes

Most instructors agree that new approaches to calculus call for new kinds of quiz and examination questions. We have included a sample of the kinds of quiz and examination questions that we have used at Ithaca College in conjunction with our courses that are based on the materials in *Calculus: An Active Approach with Projects*. This collection appears on our website at `www.ithaca.edu/calculus`.

Other Issues

Finally, when we have conducted workshops on our active approach to calculus, a number of questions have arisen about how one can integrate these materials into a course. We include some of the most often asked questions and our responses in the next few sections of this introduction.

About Using Projects

Question: How can using projects improve my course?

Answer: Projects bring out both the relevance and the unity of calculus. Most of the projects are set in "realistic" situations. Most also involve more than one calculus topic, often combining topics in unexpected ways. Students need to broaden their perspective and synthesize ideas in order to complete these projects successfully.

Many of the projects have "open-ended" parts—that is, parts for which there is not a unique correct answer or approach. These questions encourage the students to brainstorm in their groups and to view mathematics as a subject with creative elements.

To complete a project, each team needs to submit a well-written report of its solution. Writing about mathematics may be a new experience for many students, but it is a valuable one. Describing the solution of a significant problem precisely in words requires a deeper understanding than most students gain from just solving many problems that are based on examples found in their notes or textbooks.

Question: How do I organize project assignments?

Answer: We believe that the students learn more calculus if they work on the projects in teams. Three students to a group seems to be a good working number, but we have used teams of four or two students. The number depends on the class size and the particular project.

We require each team to meet with the instructor soon after the project is assigned and to present an outline or similar analysis of the problem and their anticipated solution at that time. It can be helpful for the group to present an outline for time goals to finish various parts of the project. We emphasize that the project should be completed before the due date to ensure that there is sufficient time to assemble and polish the final report. This assures us that the groups have met and that they have begun to think about the problem long before it is due. For a first project, we often require a second meeting to answer questions and get an intermediate progress report.

The final report is expected to conform to guidelines we hand out at the beginning of each semester. A sample is included in Appendix B.

Question: How many projects should I use?

Answer: We have found two to four projects to be an effective choice for a fourteen-week course. We are convinced that assigning several projects provides significantly greater benefit than assigning only one project.

Question: How long do the students work on each project?

Answer: Most projects take two to three weeks for a group of students to complete. The work is done outside class and is in addition to any daily assignments that we make for the class. Our students have a project in progress much of the time.

Question: When should the first project be assigned?

Answer: We believe that it is best to assign the first project early in the course. This helps students to view the projects as an integral part of the course experience. It also helps them to develop significant problem-solving skills that they can bring to bear on other elements of the course.

Question: How should projects affect the course grade?

Answer: Substantially. We count each project as 10% of the total course grade, so that in a four-project class, 40% of the course grade is based on the projects. This is a very effective way to ensure that the students take the projects seriously and devote substantial thought and effort to them.

Question: Can I use projects without having the students work in teams?

Answer: Yes, but at a price. If students are working together to solve these problems, you can have higher expectations for the solutions—after all, they had the advantage of collaborative efforts. More important, working on a team provides effective peer pressure on students not to give up trying to solve a problem. Many employers have mentioned that the ability to work effectively as part of a team is an important skill for most jobs. For the instructor, there is the added advantage of fewer projects to grade!

Question: How do I answer questions about a project?

Answer: Just as you would any other graded assignment. Ask them what they have tried and why. Be sure the project team has discussed the question as a group and that everyone on the team has tried to answer it. Then suggest they think about certain ideas that might prove helpful. Try to steer them in appropriate directions without "giving away" solutions. If the same question is raised by a number of teams, it is sometimes helpful to discuss the issue briefly with the whole class.

Question: How much class time should I spend on projects?

Answer: We spend little or none. Our class efforts are devoted to making available to the students the tools that they will need in order to complete the projects successfully. These include the standard tools of calculus, new approaches that can be found in the activities, and practice (again through the activities) in modeling and other problem-solving strategies. One might spend a little class time *after* a project is completed, "debriefing" and discussing how the project leads to or illustrates general concepts.

Question: How do I grade projects?

Answer: That's a good question. In our courses, the team receives a single grade on the group parts of the project. In some cases—for example, "Designing a Detector"—that is the entire project; in other cases—for example, "Spread of a Disease"—there are individual parts that receive a grade separate from the team grade.

It is a good idea to think about how you are going to assign grades before you make the assignment. After you collect the reports, scan all the solutions before you begin grading, and then make a firm decision, again before you begin grading, about how you will determine the grade.

Because there are significant written reports involved, some credit should be allocated to the manner of presentation. Derivations should be explained, graphs and other figures should be labeled, and language should be used correctly.

You need to observe whether they answered the questions posed and how well. Be prepared to reward creative ideas.

The reports should improve throughout the term. When you return first projects, indicate how presentation and reasoning can be improved for future reports. One way to do this is to give out a "collage" of good solution pieces.

Questions About Using Student Groups

Question: How do I form groups? What do I do about students who prefer to work alone?

Answer: For projects, we assign students to groups. The assignments are often based on where people live—for example, students who live in the same dorm will have a good chance of finding convenient ways to meet—or on similarity of schedules. We usually do not ask students who they would like to work with.

We usually change the groups at least once during the semester. This gives the students the opportunity to get to know several others in the class well and gives us the chance to "fix" problem groups.

Sometimes we have project groups work on an activity in class, where we can observe them and make suggestions about procedure and group dynamics.

Occasionally a student expresses the wish to work alone on the projects. Usually, we assign such a student to a group for the first project but permit her or him to work independently on at least one subsequent project.

Question: What do I do about students who don't carry their share of the load?

Answer: There are a number of techniques that we have found effective. First, you should be sure that some of the projects involve individually graded parts. These are usually the data-generating portions of the projects. This assures that a student who is not doing the work will not receive precisely the same grades as those who are.

Second, include a project-related question on the next quiz or exam.

Finally, you can adjust the group membership for subsequent projects. A method that we have found effective is to put the non-workers together in a group. This solution is not intended to be strictly punitive—often such students were simply quiet or intimidated by other group members. Putting such students together in a team often draws them out. This makes the other students feel better and often has the happy result that the members of this new group become motivated to do a good job on the next project.

A related question, of course, is how one identifies students who are not carrying their fair share of the load. One method we have found effective is to require students to submit confidential peer evaluations of all other team members at the end of each project.

Questions About Using Activities

Question: How can using activities improve my course?

Answer: Activities involve the students in their own learning. Both attendance and participation improve. Students who have used activities regularly in class tend to make better comments and ask more significant questions about course material than did those in our more traditional classes.

The activities in *Calculus: An Active Approach with Projects* have several purposes. They introduce new calculus topics, often in a guided discovery format. This reduces the amount of formal presentation that the instructor must do. They help the students become better modelers and problem solvers. This helps the students complete projects successfully and also helps them solve shorter problems. Students who have experienced activities and projects do not regard "word problems" with dismay.

Question: How do you do an activity in class?

Answer: Each activity is different. Some involve several pieces of student work, each followed by the instructor drawing out solutions and helping with summaries. Others are done more independently by the students. Some are intended to be done outside class as follow-ups to classroom work.

Some of the activities (for example, "Gotcha") are short open-ended problems. Others (for example, "Fundamental Theorem of Calculus") provide a road map to some new calculus concept.

While the students are working, you should watch and listen to some of their discussions. You can give some guidance, but don't be too quick to show them an answer. Sometimes students will help students in another group, and we don't discourage this.

We have found that it is important that the students see the solutions and some sort of summary at the end of each activity. This immediate feedback solidifies their understanding.

Question: Do activities affect the course grade?

Answer: We usually avoid grading activities, preferring that the students feel free to use them to explore calculus ideas without thinking about how the explorations will affect their course grades. However, exams and quizzes do include questions related to the ideas they get from doing the activities.

Question: How long does each activity take?

Answer: A few activities require an entire fifty-minute class period, but many of them are much shorter. Some require as little as five minutes. A discussion of each activity appears in the "Instructor Notes for Activities." These discussions include estimates of how much time each activity requires.

Questions About Course Organization and Content

Question: How do activities and projects work together?

Answer: Both activities and projects help the students learn modeling and other problem-solving skills, and both force them to be active learners. Working on activities is good preparation for working on projects. Some of the same approaches are appropriate, although the problems in the activities are shorter and done in a more guided setting (the classroom). Working on projects helps the students to become self-starters, which helps them with their work on the activities.

Question: How do you teach problem solving?

Answer: We teach the students to break large problems into successively simpler pieces until the pieces are such that they can find solutions, then to reassemble the pieces into a solution of the original problem. We use this approach in working on problems and activities in class and expect the students to carry the ideas over to the solution of the large problems in their projects.

We also teach them to experiment with special cases, think about similar problems they have seen, make approximations, and use graphs to gain insights about the problem.

Question: How will I have time for activities?

Answer: We have found that incorporating activities into our classes does not result in time pressure. One reason is that many of the activities tend to be fairly short and can be done by the students as they settle in at the start of the class. The other is that the activities actually serve as a means of developing the course material. The student-involved approach replaces some of the formal presentation that we were accustomed to do—and does a more effective job. When the students learn through the activities, they absorb the material better and retain it longer.

Question: How do I present new material?

Answer: Quite a bit of new material comes from the activities, and still more can be introduced in projects. Furthermore, the project/activity approach does not preclude including some "traditional" kinds of presentations. We have found that using activities predisposes our students to participate more fully in even the more traditional classes, resulting in more lively classes and more give and take between students and instructor.

Question: How do quizzes and exams relate to activities and projects?

Answer: We include ideas first encountered in activities and projects on the quizzes and exams. As we mentioned above, this is one way to ensure some degree of participation in project work by all members of the project groups. We also feel free to ask non-standard questions—questions that do not correspond to any template the students have seen—on exams. Such questions are in the spirit of original problem solving that we emphasize through the activities and projects. Some sample questions are on our website at `www.ithaca.edu/calculus`.

Question: My students need to have good computational skills. How will they learn to perform computations?

Answer: Much as they always did—through daily homework assignments. There is a difference, however. In the traditional course, students come to believe that calculus *is* computations. In the new course, students see the subject itself—its unity and applications—and regard routine computational problems as an easier (and less interesting) part of the course.

We also still include some rote computations on exams and quizzes, but these are not the main focus. They are necessary tools that the students need to learn.

Question: How do you teach the logical and theoretical aspects of calculus?

Answer: While the theoretical side of calculus is taught less formally than in more traditional models, the emphasis in our courses is on conceptual understanding. We believe this approach lays a more solid foundation for subsequent study than rote memorization of definitions and theorems.

We still teach the formal definition of continuity, the derivative and the definite integral, and significant theorems (the intermediate value theorem, mean value theorem, fundamental theorems, etc.). We emphasize critical thinking about these concepts and understanding their meaning.

Question: Will I be able to cover all of the material?

Answer: We do. The activities and projects serve the same purposes as, for example, several days studying "word problems" and do the job more effectively.

Question: What is the role of technology?

Answer: Several of the activities presuppose the use of either a computer or a graphing calculator. Some of the projects are greatly simplified if some computational device is available to help with the calculations and graphs. The "Instructor Notes" list what, if anything, is needed for any particular activity or project. We do not prescribe any particular choice of technology, however. We have always described our materials as *technology independent*. This means that most of the activities and projects require no technology at all, and the few that do require a computer or graphing calculator are presented in such a way that the instructor using the material can choose whatever implementation is available.

Unifying Threads

In our work on calculus, we have identified a number of unifying threads that typically run through a successful first-year course. We have noted with interest that a number of other groups of mathematicians working on revitalizing calculus have identified similar themes. We label our threads **graphical calculus**, **distance and velocity**, **multiple representation of functions**, **modeling**, and **approximation and estimation**. We give a brief description of each thread below. We describe these themes as "threads" because they are woven throughout the course, and serve to bind it together into a unified whole. Most of the projects and many of the activities contain elements from more than one of the threads. In Appendix C we have included guides to the relationships between the projects and activities in the volume and the corresponding threads.

Threads in first-year calculus

- **Graphical calculus.** We use graphs as important examples of functions from the start of the first semester of calculus. Many of the important concepts of calculus are presented using graphs of functions (with no formula given) during the first two weeks of the course. These concepts include the slope of a function at a point, increasing and decreasing functions, concavity, continuity, extrema, and the fundamental theorem of calculus, as well as functions paired with their derivatives in the form of graph/rate-graph pairs. We continue to use many examples of functions represented by graphs throughout the year.

- **Distance and velocity.** While graphing (slopes and areas) provides one convenient example of derivatives and integrals, distance and velocity provides another. Velocity is one of the few examples of a rate that students have experienced, so students are able to understand it more intuitively than other examples. It is easy for the instructor to model velocity by simply moving around the classroom. Over the course of two or three days, we are able to introduce the important concepts of calculus by discussing average velocity, instantaneous velocity, and approximations to distance traveled. It is also easy to motivate the first derivative test, for example, by noting that the distance is maximized (locally) "when you turn around"; i.e., when velocity changes from positive to negative. Of course, we don't necessarily refer to derivatives the first day, but when we get there, this result looks familiar.

- **Multiple representations of functions.** We emphasize the notion of a function stressing that there are several representations for a function: graphical, numerical (a table of data), algebraic (a formula or expression), physical, theoretical, and written or verbal. This is important, because students need to be able to apply techniques they learn to functions presented in any of these formats. It is not enough for students to simply learn to manipulate symbols.

  This idea also helps students see the unity of calculus. For instance finding the area under a graph, finding the antiderivative of a function, and computing a Riemann sum from a table of values are all examples of the concept of integration. We emphasize the interplay of all representations. If we have a result for a function, some typical questions from the instructor are "What if you were given a graph? Then what does this result tell you? What if you were given a table of values? What does this result mean? Describe this situation in English."

- **Modeling.** Modeling begins early in the course and continues throughout the first year. Projects and classroom examples typically begin with some "real" application, which must be translated into a mathematical model.

  Modeling is introduced early by way of classroom activities. Graphical relationships are the first instance of modeling. For example, in class students are asked to create and analyze a graphical model of the height of a flag from the ground as a function of time as the flag is being raised.

  The students use modeling techniques on the projects. For example, in a Calculus 1 project, students are asked to model the motion for a detector that is guarding a hallway against intruders.

  This emphasis on modeling in class and in the projects seems to have strengthened the students' belief that it is possible to construct functions that model even complex situations and that the concepts presented in calculus are valuable tools in this process. Furthermore, it seems to have significantly reduced the students' fear of word problems.

- **Approximation and estimation.** Approximation and estimation are stressed as fundamental and unifying concepts in calculus in a theoretical sense (they appear in the concepts of the limit of a function, the derivative, the integral, and the convergence of sequences and series). Moreover, these notions are stressed as appropriate solutions to specific problems. For example, students are asked in class in a group setting to estimate the slope of a graph of a function at a point using only the graph of the function on graph paper and a ruler. They then compare answers and discuss the question of who has the "correct" answer. Similarly, they experience approximation by applying at least one numerical method to the problem of root finding to a given degree of accuracy and they study at least one method of numerical integration, complete with an error estimate.

  Students are also asked to compute Riemann sums and finite differences from tables of values and solve certain problems using these numerical estimates. Indeed, the students view some modeling experiences as estimation or approximation of a "real" situation. They also see Taylor polynomials and/or other functions obtained from curve fitting serving as approximations to the original function, which may be used in place of the original function under certain conditions. Finally, where appropriate, estimation is stressed in connection with common sense. That is, students are asked if their answers to problems make sense. In this way, "mathematical" common sense is both developed and reinforced.

Introducing students to these materials

Finally, we think it is important that students understand the spirit of the materials in this collection. So we include here the introduction for students that appeared in the original edition.

To the Student

This is a book of activities and projects for calculus. It is designed to help you to understand the basic concepts of calculus and to become a good problem solver.

To use these materials successfully, you should approach them with an open mind and a lot of optimism. The activities are relatively short calculus problems or explorations. Most of them will not look like problems you have seen before. All of them are problems on which almost any first-year calculus student can make significant progress.

Working through the activities should help you understand the basic concepts of calculus and how calculus serves as a tool for understanding the world. The activities will also help you learn ways to approach unfamiliar problems and to make progress in solving them.

The projects are larger problems which will require considerable effort to solve. Many of the projects involve several different calculus ideas, so be prepared to draw on all your mathematical background. Many are also "open-ended" problems. That means that there isn't necessarily just one correct solution. Your ideas for these problems will need to be supported by well thought out explanations that will be convincing to others.

Your instructor will decide just how these materials will be used in your particular course. We hope this book will help make calculus interesting, challenging, and meaningful to you.

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